§ Lecture 9.0

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Lemma [Rationals between integers].

Let $x \in \mathbb{Q}$. Then there exists a unique integer n such that

$$n \le x < n+1.$$

Proof: Let $x = \frac{p}{q}$, where $p, q \in \mathbb{Z}$ and $q \neq 0$.

Case 1: Let $x \geq 0$, then $p \geq 0$ and $q > 0$.

In this case p is a natural number. Then there exist natural numbers n, r such that

$$p = qn + r \quad , \text{where} \quad 0 \le r < q. \quad (\text{Division algorithm})$$

Then $\frac{p}{q} = n + \frac{r}{q}$

Since $\frac{r}{q} \geq 0 \Rightarrow \frac{p}{q} \geq n$

Since $\frac{r}{q} < 1 \Rightarrow \frac{p}{q} < n+1.$

That is there exists a natural number (hence integer) n s.t.

$$n \le x < n+1.$$

Case 2: $x < 0$. Then $-x > 0$. Then there exists $n \in \mathbb{N}$ s.t.

$$n \le -x < n+1 \quad \text{or} \quad -n-1 < x \le -n$$

This is incorrect.

Recall that if $x<y \Rightarrow x+1 \le y$ for $x, y \in \mathbb{N}$.

$\Rightarrow -x < n+1 \Rightarrow -x+1 \le n+1$ or $-x \le n$.

And $n-1 < n$.

Then we have $n \le -x < n+1 \Leftrightarrow n-1 < -x \le n$

$\Rightarrow -n \le x < -n+1$

Take $m = -n \in \mathbb{Z}$, then

$m \le x < m+1$

Case 1: $x = -n$, then take $m = -n$. Then

$x = m < m+1 \Rightarrow m \le x < m+1.$

Case 2. $x < -n$. Take $m = -n-1$

Then $x < -n = m+1$

and $x > -n-1 = m. \Rightarrow x \geq m$ (Strict inequality implies non strict)

Thus $m \le x < m+1$

Note that $x > -n-1$ in both above cases.

Thus in both cases, there exists an integer n s.t.

$$n \le x < n+1.$$

Uniqueness: Let m, n s.t. $m \neq n$ are such that

$$n \le x < n+1 \;\; \& \;\; m \le x < m+1.$$

WLOG take $m < n \Rightarrow m+1 \le n.$

Then $x < m+1 \le n \le x$

$\Rightarrow x < x$ Contradiction $\Rightarrow m = n.$

This n is called floor of x, i.e. $\lfloor x \rfloor = n.$

Lemma [Rationals between rationals]:-

Let $x, y \in \mathbb{Q}$ and $x < y$. Then there exists a rational z s.t.

$$x < z < y.$$

Proof: Let $z = \frac{x+y}{2}$.

① $x < y \Rightarrow \frac{x}{2} < \frac{y}{2} \Rightarrow z < y$

② $x < y \Rightarrow \frac{x}{2} < \frac{y}{2} \Rightarrow z > x$

$\Rightarrow x < z < y.$

$z = ax + (1-a)y \qquad a \in (0,1).$

Then $z < x$ & $z > y$ using $x < y$.

This completes the proof.

Lemma There doesn't exist any $x \in \mathbb{Q}$ s.t. $x^2 = 2.$

Proof: For the sake of contradiction assume that there exists $x \in \mathbb{Q}$ s.t. $x^2 = 2.$

Note that $x \neq 0$. WLOG we can take $x > 0$. (In other case we can take $-x$ and $x^2 = (-x)^2$).

Let $x = \frac{p}{q}$, where $p, q \in \mathbb{N} - \{0\}.$

Then $x^2 = 2 \Rightarrow p^2 = 2q^2.$

Definitions:

Even and odd natural numbers: A natural number p is said to be

(i) even if $p = 2k$ for some $k \in \mathbb{N}$.

(ii) odd if $p = 2k+1$ for some $k \in \mathbb{N}$.

Claim: Every natural number is either even or odd.

Proof: Let $p = 2k_1$ for some $k_1 \in \mathbb{N}$ and

$p = 2k_2 + 1$ " " $k_2 \in \mathbb{N}$.

$\Rightarrow 2k_1 = 2k_2 + 1$

$\Rightarrow 2(k_1 - k_2) = 1$ $\quad$ $(k_1 - k_2)$ is an integer while $1/2$ is not an integer.

ⓐ $k_1 - k_2 = 0 \Rightarrow 1 = 0$

ⓑ $k_1 - k_2 \geq 1 \Rightarrow 1 \geq 2$

ⓒ $k_1 - k_2 \le -1 \Rightarrow 1 \le -2$

Impossible.

Claim: If p is odd then p^2 is odd.

Proof. Let $p = 2k+1 \Rightarrow p^2 = 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$

$= 2k' + 1$ (Because $2k^2 + 2k \in \mathbb{N}$).

If p is odd then $p^2 = 2q$ cannot be true because p^2 is odd.

Thus p is an even natural number.

Hence $p = 2k$ for some $k \in \mathbb{N} - \{0\}.$

Then $p^2 = 2q^2 \Rightarrow 4k^2 = 2q^2 \Rightarrow q^2 = 2k^2.$

So we started from (p, q) with $p^2 = 2q^2 \Rightarrow (q, k)$ with $q^2 = 2k^2$

So we can have ($p_0 = p$, $q_0 = q$; $p_1 = q$, $q_1 = k$, $p_2 = k$, $p_3 = \cdots$)

$(p_0, q_0) \rightarrow (p_1, q_1) \rightarrow (p_2, q_2) \cdots \rightarrow \cdots$

Note that $p^2 = 2q^2 \Rightarrow q < p.$ $\left(\begin{array}{l} q \neq p \\ q > p \Rightarrow 2q^2 > 2p^2 > p^2 \end{array} \right)$ Impossible.

Since $q < p$, by construction we have $p_0 > p_1 > p_2 \cdots$

s.t. $p_i^2 = 2q_i^2$. (This is impossible as we should be reaching to $p_n = 0$).

[Principle of infinite descent].

Exercise: A sequence $a_0, a_1, \cdots \equiv (a_n)_{n \in \mathbb{N}}$ is said to be infinite descent if we have $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$.

Principle of infinite descent: It is impossible to have a sequence of natural numbers which is in infinite descent.

Proof: For contradiction assume that there exists a sequence $(a_n)_{n \in \mathbb{N}}$ of natural numbers s.t. $a_n > a_{n+1}$ $\forall n \in \mathbb{N}$.

Claim: For every $k \in \mathbb{N}$, we have $a_n \geq k$ $\forall n \in \mathbb{N}$.

Proof: Induction on k. $P(k) =$ " $a_n \geq k$ $\forall n \in \mathbb{N}$ "

Base Case: $k = 0$, $P(0) =$ " $a_n \geq 0$ $\forall n \in \mathbb{N}$".

This is true because $a_n \in \mathbb{N}$ $\forall n$.

Inductive step: Assume $P(k)$ is true. That $a_n \geq k$ $\forall n \in \mathbb{N}$.

Fix any n. Then by infinite descent

$a_n > a_{n+1} \geq k$

$\Rightarrow a_n \geq k+1$

Since n is arbitrary we have $a_n \geq k+1$ $\forall n \in \mathbb{N}$. This completes the proof.

Now apply this claim to $n = 0$ and $k = a_0 + 1$, we have

$a_0 \geq a_0 + 1$ This is impossible.

Thus we can't have infinite descent.

$a_0 \quad a_1 \quad a_2 \cdots a_n \quad a_{n+1} \quad a_{n+2} \cdots$

≥ 1

≥ 2

≥ 3

$a_n \geq k$ $\forall n \in \mathbb{N}$ $\forall k \in \mathbb{N}$.

Lemma: For every rational number $\varepsilon > 0$, $\exists$ a nonnegative rational number x s.t. $x^2 < 2 < (x+\varepsilon)^2.$

Proof: Let $\varepsilon > 0$ be a rational number. Fix it.

For contradiction assume that there is no nonnegative rational number x s.t.

$$x^2 < 2 < (x+\varepsilon)^2.$$

What does this mean?

$\forall x \in \mathbb{Q}$: $x \geq 0$ and $x^2 < 2 \Rightarrow (x+\varepsilon)^2 < 2.$ Equality

(Note that $\forall x \in \mathbb{Q}$: $x \geq 0$ and $(x+\varepsilon)^2 > 2 \Rightarrow x^2 > 2$) Equivalent. We use above.

Note that $\varepsilon > 0$, then $2\varepsilon, 3\varepsilon, \cdots$ all are positive.

$\Rightarrow$ We have $\varepsilon^2 < 2$, $(2\varepsilon)^2 < 2, \cdots$

In fact, by induction $(n\varepsilon)^2 < 2$ $\forall n \in \mathbb{N}.$

But from $n \le x < n+1$, we can find an integer n_0

s.t. $n_0 > 2/\varepsilon. \Rightarrow n_0 \varepsilon > 2 \Rightarrow (n_0 \varepsilon)^2 > 4 > 2.$

But since $(n_0 \varepsilon)^2 < 2$, we have a contradiction.

Thus $\forall \varepsilon > 0$, $\exists x \in \mathbb{Q}$, $x \geq 0$ s.t. $x^2 < 2 < (x+\varepsilon)^2.$

Story: Rationals can't be equal to $\sqrt{2}$ but they can reach as close to $\sqrt{2}$ as one wants.

This motivates introducing $\sqrt{2}$ as "some sort of limit of rationals".