§ Lecture 8.0

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Negation of integers: If $a-b$ is an integer, the negation $-(a-b)$ is defined as the integer $b-a$.

In particular, $n = n-0$ then $-n = 0-n$.

Trichotomy of integers: Let x be an integer. Then exactly one of the following statements is true:

(i) $x=0$. (ii) $x = n-0$ for some positive natural number n.

(iii) $x = 0-n$ " " " " n.

Subtraction: $x, y \in \mathbb{Z}$ then $x-y = x+(-y)$.

For natural numbers a, b : $a-b = a+(-b)$

$$= (a-0)+(0-b)$$

$$= a-b$$

Thus $a-b$ and $a-b$ are the same things.

Ordering of integers: Let x, y be integers.

(i) $x \geq y$ iff $x = y+a$, $a \in \mathbb{N}$.

(ii) $x > y$ iff $x \geq y$ and $x \neq y$.

Claim: $a > b \Rightarrow -b > -a$.

Proof: $a>b \Rightarrow a = b+m$, $m \neq 0$, $m \in \mathbb{N}$.

$\Rightarrow -a = -(b+m)$

$\Rightarrow -a+m = -b$

$\Rightarrow -b > -a$.

The rationals: A rational number is an expression of the form $a//b$, where $a, b \in \mathbb{Z}$ and $b \neq 0$. Further, $a//b = c//d \Leftrightarrow ad = bc$.

The set of rationals is denoted by $\mathbb{Q}$.

Addition: $a//b + c//d = (ad+bc)//bd$, $b \neq 0, d \neq 0$. $a, b, c, d \in \mathbb{Z}$

Product: $(a//b)(c//d) = (ac)//(bd)$

Negation: $-(a//b) = (-a)//b$.

Remark. $a//1$ behaves like integer a.

$a//1 + b//1 = (a+b)//1$

$(a//1)(b//1) = (ab)//1$

$-(a//1) = (-a)//1$

Thus we can identify integers as rationals $a//1$.

Note: $a//b = 0$ iff $a = 0$.

Reciprocal: Let $x = a//b \in \mathbb{Q}$ and $a \neq 0$, $b \neq 0$. Then reciprocal of x is defined as

$$x^{-1} = b//a.$$

Claim: $xx^{-1} = 1 = xx^{-1}$.

Proof: Let $x = a//b$, $a \neq 0$, $b \neq 0$.

$$xx^{-1} = (a//b)(b//a) = (ab)//(ab) = 1//1 = 1.$$

Similarly $x^{-1}x = 1$.

Quotient x/y:

$$x/y = x \cdot y^{-1}.$$

Claim: $x//y = x/y$ for integers x, y with $y \neq 0$.

Proof: $x/y = x \times y^{-1} = (x//1) \times (1//y)$

$$= x//y.$$

Thus we can use x/y instead of $x//y$.

Positive and negative rational numbers:

x is positive $\Leftrightarrow$ $\exists$ positive integers a, b s.t. $a//b = x$.

x is negative $\Leftrightarrow$ $x = -y$ for some positive rational y.

Ordering of rationals:

(i) $x > y$ $\Leftrightarrow$ $x-y$ is a positive rational.

(ii) $x \geq y$ $\Leftrightarrow$ Either $x > y$ or $x = y$.

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Absolute value: Let $x \in \mathbb{Q}$. Then absolute value of x is defined as

$$|x| := \begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0. \end{cases}$$

Claim: (i) $|x| \geq 0$. $|x|=0 \iff x=0$.

(ii) $|x+y| \leq |x|+|y|$

(iii) $-y < x < y \iff |x| < y$. Also $-|x| \leq x \leq |x|$.

(iv) $|xy| = |x|\,|y|$.

Proof: (i) Note that $x>0$ or $x<0$ or $x=0$.

Case 1. $x>0 \Rightarrow |x| = x > 0$

Case 2. $x<0 \Rightarrow |x| = -x > 0$

Case 3: $x=0 \Rightarrow |x|=0$

$\Rightarrow |x| \geq 0$.

$*$ $|x|=0 \iff x=0$.

Proof: $\Rightarrow$ $x=0 \Rightarrow |x|=0$.

$\Leftarrow$ $|x|=0 \Rightarrow x=0 \iff x\neq 0 \Rightarrow |x| \neq 0$.

If $x \neq 0$ then either $x>0$ or $x<0$ in both cases $|x|>0$.

(iii) $\Rightarrow$ Suppose $-y \leq x \leq y$

Case 1: $x \geq 0$ then $|x| = x \leq y$.

Case 2: $x<0$ then $|x| = -x \leq y$

Thus $|x| \leq y$.

$\Leftarrow$ Suppose $|x| \leq y$.

Since $|x| \geq 0$, we have $y \geq |x| \geq 0$.

$\Rightarrow -y \leq 0 \leq y$.

Case 1: If $x>0$ then $x = |x| \leq y$

& $-y \leq 0 \leq x$.

$\Rightarrow -y \leq x \leq y$.

Case 2: If $x<0$ then $-x = |x| \leq y$

$\Rightarrow -y \leq x$.

& $x < 0 \leq y$

$\Rightarrow -y \leq x \leq y$.

Taking $y = |x|$, and noting $|x| \leq |x|$.

we have $-|x| \leq x \leq |x|$.

(ii) $|x+y| \leq |x|+|y|$

Proof: Note that $-|x| \leq x \leq |x|$

$-|y| \leq y \leq |y|$

$\Rightarrow -(|x|+|y|) \leq x+y \leq |x|+|y|$

$\Rightarrow |x+y| \leq |x|+|y|$

(iv) $|xy| = |x|\,|y|$

Proof: If $x=0$ or $y=0$ then LHS $=0=$ RHS.

There are 4 remaining cases.

(i) $x>0$, $y>0$. Here $xy>0 \Rightarrow xy = |xy|$

$|x| = x$, $|y| = y \Rightarrow |x|\,|y| = xy$.

$\Rightarrow |xy| = |x|\,|y|$.

(ii) $x>0$, $y<0$. Then $|xy| = -(xy) = x(-y) = |x|\,|y|$.

Similarly other cases.

In particular, take $y=-1$, then $|xy| = |-x|$

$|x|\,|y| = |x|$. $\Rightarrow |-x| = |x|$.

Distance: Let $x, y \in \mathbb{Q}$. Then distance between x, y is defined as

$$d(x,y) = |x-y|.$$

It satisfies:

(i) $d(x,y) \geq 0$. $d(x,y) = 0 \iff x=y$.

Proof: $d(x,y) = |x-y| \geq 0$

and $|x-y| = 0 \iff x-y=0$ or $x=y$.

(ii) $d(x,y) \leq d(x,z) + d(z,y)$

Proof: $d(x,y) = |x-y|$

$= |x-z+z-y|$

$\leq |x-z| + |z-y|$

$= d(x,z) + d(z,y)$.

(iii) $d(x,y) = d(y,x)$

Proof: $d(x,y) = |x-y| = |y-x| = d(y,x)$.

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Definition (ε-closeness): Let $\varepsilon > 0$ be a rational number, let $x, y \in \mathbb{Q}$. We say that y is ε-close to x iff $d(y,x) \le \varepsilon$.

Claim: (i) $d(x,y) \le \varepsilon$ for all $\varepsilon > 0$ is true iff $x = y$.

Proof: $\Rightarrow$ If $x = y$ then $d(x,y) = 0 < \varepsilon \quad \forall \varepsilon > 0$.

$\Leftarrow$ Assume $d(x,y) \le \varepsilon \quad \forall \varepsilon > 0$. Then we want to prove that $x = y$.

Contrapositive: $x \ne y \Rightarrow \exists\, \varepsilon_0 > 0$ s.t. $d(x,y) > \varepsilon$.

Proof: Since $x \ne y$, let $x > y$ and $x - y = a > 0$.

Choose $\varepsilon_0 := a - \frac{1}{n+1} < a$ for any $n \in \mathbb{N}$.

Then $|x-y| = x - y = a > \varepsilon_0$.

This concludes the proof.

(ii) Let $|x-y| \le \varepsilon$, $|z-w| \le \delta$, then $|x+z-(y+w)| \le \varepsilon + \delta$.

Proof: $|x+z-y-w| \le |x-y| + |z-w| \le (\varepsilon + \delta)$.

(iii) $d(x,y) \le \varepsilon$, $d(z,w) \le \delta$, then $d(xz, yw) \le (\varepsilon|z| + \delta|x| + \varepsilon\delta)$.

Proof: $d(xz, yw) = |xz - yw|$

$$= |xz - yz + yz - yw|$$

$$\le |z|\,|x-y| + |y|\,|z-w|$$

$$\le |z|\,\varepsilon + |y|\,\delta$$

Hmm!!

Different trick: Call $x - y = a \Rightarrow |a| \le \varepsilon$

$z - w = b \Rightarrow |b| \le \delta$.

$$\Rightarrow xz = (a+y)(b+w)$$

$$xz - yw = ab + aw + yb$$

$$|xz - yw| \le |ab| + |a|\,|w| + |y|\,|b|$$

$$\le \varepsilon\delta + \varepsilon|w| + \delta|y|.$$

Hmm! Take $y - x = a$

$w - z = b$

$$\Rightarrow (xz - yw) = xz - (a+x)(b+z)$$

$$= -ab - az - bx$$

$$\Rightarrow d(xz, yw) \le |a|\,|b| + |a|\,|z| + |b|\,|x|$$

$$\le \varepsilon\delta + |z|\varepsilon + |x|\delta.$$

All of them are correct.

Exponentiation: Let $x \in \mathbb{Q}$. Define $x^0 = 1$ $(0^0 = 1)$.

Assume x^n is defined for some natural number n. Then $x^{n+1} = x^n \times x$.

Claim: $x^n \cdot x^m = x^{n+m} = x^m \cdot x^n$. $(x^n)^m = x^{nm}$.

Proof: Induction on n. Fix m.

Base case: $n = 0$ $\quad x^0 \cdot x^m = x^{0+m} \Rightarrow x^m = x^m$.

Inductive step: $x^n \cdot x^m = x^{n+m}$

Then $x^{n+1} \cdot x^m = x^n \cdot x^m \cdot x$

$$= x^{n+m} \cdot x$$

$$= x^{m+n+1}$$

Similarly for $(x^n)^m = x^{nm}$.

Exponentiation to a negative number:

$x^{-n} = 1/x^n \qquad n > 0.$