§ Lecture 7.0

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Claim: Empty set has cardinality zero.

Proof: zero cardinality of ϕ means that there is a bijection from ϕ to $\{i : 1 \le i \le 0\} = \phi$.

The empty function is such a bijective mapping.

Empty function: $f : \phi \to \phi$

This needs no specification as there are no inputs.

① $\forall x, x' \in \phi \quad f(x) = f(x') \Rightarrow x = x'$. [Vacuously true as there are no x, x' in ϕ]

② $\forall y \in \phi$ there is an $x \in \phi$ s.t. $y = f(x)$. [Vacuously true as there are no y in ϕ]

Thus empty set has cardinality zero.

Remark: Empty function is a promise that you never have to hold as there will be no situation in which you will be tested.

" $\forall x \in \phi$, $y \in Y$ then $P(x,y)$ be "whatever" property such that there is exactly one y for each x. Then $f(x) = y$ if $P(x, f(x))$ is true."

|| No such x ever arise ||

Finite sets: A set is finite iff it has cardinality $n \in \mathbb{N}$ otherwise it is called infinite. If X is a finite set then let us define

$|X|$ = cardinality of X.

Theorem: The set $\mathbb{N}$ of natural numbers is infinite.

Proof: Let $\mathbb{N}$ is finite and $|\mathbb{N}| = n$, where $n \in \mathbb{N}$. Then there is a bijection $f : [n] \to \mathbb{N}$. [Recall $[n] = \{i \in \mathbb{N} : 1 \le i \le n\}$]

Claim: For every function $f : [n] \to \mathbb{N}$ $\exists$ $M \in \mathbb{N}$ s.t. $f(i) \le M$ $\forall i \in [n]$.

Proof: By induction.

Base case: $n = 0$, the function $f : \phi \to \mathbb{N}$ is unique [$[0] = \phi$].

Since the domain is empty we can take $M = 0$. as requirement $f(i) \le M$ is vacuous as there is no i.

Inductive step: Assume the claim is true for some n. ie.

$\exists M_0$ s.t. $f(i) \le M_0$ $\forall i \in [n]$. $\forall f$.

Now let $g : [n+1] \to \mathbb{N}$ be a function.

Define a new function $h : [n] \to \mathbb{N}$ as

$h(i) := g(i) \quad \forall i \in [n]$

Then $h(i) \le M_0 \quad \forall i \in [n]$

Define $M = \max\{M_0, g(n+1)\}$

$\Rightarrow M_0, g(n+1) \le M$.

$\Rightarrow g(i) \le M \quad \forall i \in [n+1]$

Since $g : [n+1] \to \mathbb{N}$ is arbitrary, the claim is true for $n+1$.

Thus, the bijective map $f : [n] \to \mathbb{N}$ is bounded. That is $\forall i \in [n]$, $f(i) \le M$.

Then since $M \in \mathbb{N} \Rightarrow M+1 \in \mathbb{N}$. And since $M+1 > M$, there is no $i \in [n]$ s.t. $f(i) = M+1$. $\Rightarrow$ f can't be surjective.

[Contradiction]

$\Rightarrow$ $\mathbb{N}$ is infinite.

§ Lecture 7.1

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Natural numbers to integers:

Integers: An integer is an expression of the form $a-b$, where $a, b \in \mathbb{N}$.

[$a-b$ is a single expression and an integer is defined via two natural numbers]

$-$ = This is not minus sign.

Two integers are called equal, i.e. $a-b = c-d$, iff $a+d = c+b$.

Let us denote set of integers as $\mathbb{Z}$.

Remark: Let $\sim$ be a relation on $\mathbb{N} \times \mathbb{N}$ defined by

$(a,b) \sim (c,d) \iff a+d = c+b$.

This is an equivalence relation — Reflexive, symmetric, transitive

Then $a-b = \{(c,d) \in \mathbb{N}\times\mathbb{N} : (a,b) \sim (c,d)\}$

[All pairs equivalent to (a,b)]

Ex. $3-5 = 1-3 = 5-7 = \cdots\cdots$ All are equal.

But 3 is not an integer as it is not of the form $a-b$.

Def (Sum and product of integers): The sum of two integers $a-b$, $c-d$ is defined as

$(a-b) + (c-d) = (a+c) - (b+d)$.

Product $(a-b) \times (c-d) = (ac+bd) - (ad+bc)$.

These are well defined. (Substitution axiom follows)

Special integers: $(n-0)$:

① $(n-0) + (m-0) = (n+m) - 0$

② $(n-0) \times (m-0) = (nm) - 0$

③ $(n-0) = (m-0) \iff n = m$.

Isomorphism from $(\mathbb{N}-0) \to \mathbb{N}$.

So we identify natural numbers n with integers of the form $n-0$. That is $n \equiv n-0$.

$0 = 0-0$; $1 = 1-0 \cdots$;

Def. (Negation of integers): If $(a-b)$ is an integer, we define negation $-(a-b)$ to be the integer $b-a$.

Then $n = n-0 \Rightarrow -n = 0-n$.

Lemma (Trichotomy of integers): Let x be an integer. Then exactly one of following statements is true.

(i) x is zero. (ii) x is a positive natural number.

(iii) x is negation of a positive natural number.

Algebra of integers: ① $x+y = y+x$

$\vdots$

$x + (-x) = 0$. ← New rule

rest are same as natural numbers.

Definition (Subtraction):- Let $x, y \in \mathbb{Z}$ then subtracting y from x denoted as $x - y = x + (-y)$

↑ Negation of integer.

$a-b$ and $a-b$ are the same expressions:

Let $a, b \in \mathbb{N}$

$$\begin{aligned} a-b &= a + (-b) \\ &= (a-0) + (0-b) \\ &= (a+0) - (0+b) \\ &= a-b. \end{aligned}$$

Ordering of integers: Let $m, n \in \mathbb{Z}$, we say that $n \geq m$ or $m \leq n$ iff $n = m+a$ for some $a \in \mathbb{N}$. We say $n > m$ or $m < n$ iff $n \geq m$ and $n \neq m$.