§ Lecture 6.2

Monday, 31 August 2026 21:27

Cardinality of sets :-

Cardinals : One, two, three - - - [Number of elements in a set]

ordinals : First, second, third - - - [order of a sequence of objects]

Definition (Equal cardinality) :- We say that two sets X and Y have equal cardinality iff there exists a bijection $f : X \to Y$.

Remark: Let $E = \{2n : n \in \mathbb{N}\} \subseteq \mathbb{N}$. But there is a bijection $f : \mathbb{N} \to E$, $f(n) = 2n$.

So E and $\mathbb{N}$ equal cardinality while $E \subseteq \mathbb{N}$.

(Though it looks like E must have half the elements compared to $\mathbb{N}$).

Claim: Let X, Y, Z be sets. Then

1. X has equal cardinality with X. (Reflexive)
2. X has equal " " Y, then Y has equal cardinality X. (Symmetric)
3. X has equal cardinality with Y
   Y " " " " Z
   then X " " " " Z.

Definition (Cardinality): Let n be a natural number. A set X is said to have cardinality n, iff it has equal cardinality with $\{ i \in \mathbb{N} : 1 \leq i \leq n \}$. X is said to have n elements iff it has cardinality n.

§ Lecture 6.0

Monday, 31 August 2026 5:12 PM

Axiom 10 (Power set axiom): Let X and Y be sets. Then

$\{f : f: X \to Y\}$ is a set, denoted as Y^X.

Lemma: Let X be a set. Then

$\{Y : Y \subseteq X\}$ is a set.

Proof: Recall axiom 6 (Replacement).

Let A be a set. Let $P(x,y)$ be a property of $x \in A$ and some y such that there is at most one y for each x such that $P(x,y)$ is true. Then

$\{y : P(x,y)$ is true for some $x \in A\}$ is a set.

(This axioms requires a set A, which we donot have. So

trying $\{Y : Y \subseteq X\}$ isn't correct!

Correct thing would be $\{Y : Y \subseteq X$ for some $X \in A\}$

How do we get this?

Power set axiom gives us such an A.

Let $S := \{Y : Y \subseteq X\}$

Note that since $\{0,1\}$ and X are sets.

$\{0,1\}^X = \{f : f: X \to \{0,1\}\}$ is a set from axiom 10 (Powerset).

Then for each $f \in \{0,1\}^X$ define

$A_f = \{x \in X : f(x) = 1\}$ [This is a set from axiom 5] (Specification)

Clearly $A_f \subseteq X$.

Now let us apply axiom 6, with $A \equiv \{0,1\}^X$ with property

$P(f, Y) = $ "$Y = \{x \in X : f(x) = 1\}$" for each $f \in \{0,1\}^X$.

We need to make sure that for each $f \in \{0,1\}^X$ there is at most one Y such that $P(f,Y)$ is true.

Let for a given f there are Y and Y' such that

$P(f,Y) = $ "$Y = \{x \in X : f(x) = 1\}$" is true

$P(f,Y') = $ "$Y' = \{x \in X : f(x) = 1\}$" is true.

But clearly $Y = Y'$. Thus, in fact, there is exactly one Y for each $f \in \{0,1\}^X$ s.t. $P(f,Y)$ is true.

Now axiom 6 gives:

$\{Y : P(f,Y)$ is true for some $f \in \{0,1\}^X\}$. is a set.

or

$T = \{Y : Y = \{x \in X : f(x) = 1\}$ for some $f \in \{0,1\}^X\}$ is a set.

$= \{Y : Y = A_f$ for some $f \in \{0,1\}^X\}$

$= \{A_f : f \in \{0,1\}^X\}$

Goal: $S = T$.

$\Rightarrow$ $S \subseteq T$: Let $Y \in S \Rightarrow Y \subseteq X$

Define $\forall x \in X$: $f_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{otherwise} \end{cases}$

Clearly $f_Y \in \{0,1\}^X$.

$A_{f_Y} = \{x \in X : f_Y(x) = 1\}$

$= \{x \in X : x \in Y\}$

$= X \cap Y = Y$

Thus $Y = A_{f_Y} \in \{0,1\}^X$

$\Rightarrow$ $S \subseteq T$.

$\Leftarrow$ $T \subseteq S$: Let $Y \in T \Rightarrow Y = A_f$ for some $f \in \{0,1\}^X$.

$Y = \{x \in X : f(x) = 1\} \subseteq X$

$\Rightarrow Y \in S$.

This completes the proof.

§ Lecture 6.1

Monday, 31 August 2026 19:36

Axiom 11 (Union): Let A be a set and all its elements be sets themselves. Then there exists a set $\bigcup A$ whose elements are precisely the elements of elements of A. That is

$$x \in \bigcup A \iff (x \in S \text{ for some } S \in A).$$

A consequence: If I is a set and for each $\alpha \in I$, we have some set A_α, then

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \underbrace{\{A_\alpha : \alpha \in I\}}$$

Apply axiom 6 for set I, with property

$P(\alpha, Y) =$ "$Y = A_\alpha$" (There is a single Y for each α).

Then $\{Y : Y = A_\alpha \text{ for some } \alpha \in I\}$ is a set $= \{A_\alpha : \alpha \in I\}$.

(Axiom 11).

Def: Let I be a non empty set and for each $\beta \in I$, A_β is a set.

Intersection: $\bigcap_{\alpha \in I} A_\alpha := \underbrace{\{x \in A_\beta : x \in A_\alpha \ \forall \alpha \in I\}}$ is also a set.

Axiom 5 to A_β [This set is there because I is nonempty so $\beta \in I$ and hence A_β is there]

Summary: Axioms $1 \to 7$ and $9-11$ are known as the Zermelo-Fraenkel axioms of set theory.

Cartesian product:

Ordered pair:- If x and y are any objects (possibly equal), we define the ordered pair (x,y) to be a new object, consisting of x as a first object and y as a second object. Further,

$$(x,y) = (x',y') \iff x = x' \text{ and } y = y'.$$

These are not sets. $(2,1) \neq (1,2)$

e.g. $(1,2) \neq \{1,2\}$

One can possibly derive these objects.

Attempt 1: $\{\{x\}, \{x,y\}\} = (x,y)$

One needs to show that $(x,y) = (x',y') \iff x = x'$ and $y = y'$!

① If $x = x'$, $y = y'$ then $\{\{x\}, \{x,y\}\} = \{\{x'\}, \{x',y'\}\}$.

That is $(x,y) = (x',y')$

② If $\{\{x\}, \{x,y\}\} = \{\{x'\}, \{x',y'\}\}$

Case 1: $x = y$. Then $\{\{x\}, \{x,y\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$.

Then $\{\{x'\}, \{x',y'\}\} = \{\{x\}\}$.

$\Rightarrow \{x'\} = \{x\}$ and $\{x',y'\} = \{x\}$

$\Rightarrow x' = x$ and $x' = x = y'$

$\Rightarrow x' = x = y = y'$.

Case 2: $x \neq y$

If $x' = y'$ then $\{\{x'\}, \{x',y'\}\} = \{\{x'\}\}$

Then $\{\{x\}, \{x,y\}\} = \{\{x'\}\} \Rightarrow x = y$ (contradiction)

Thus $x' \neq y'$.

$\{\{x\}, \{x,y\}\} = \{\{x'\}, \{x',y'\}\}$

$\Rightarrow \{x\} = \{x'\}$ or $\{x\} = \{x',y'\}$

$\Rightarrow x = x'$ or $x = x' = y'$ ✗

$\Rightarrow x = x'$.

Similarly $\{x,y\} = \{x'\}$ or $\{x,y\} = \{x',y'\}$

$\Rightarrow \underset{\times}{x} = y = x'$ or $y = y'$

This concludes.

Other definition could be $\{x, \{x,y\}\} = (x,y)$ [This is also good]

Both are good definitions and both are sets and hence both are objects.

Cartesian product: If X and Y are sets, then cartesian product $X \times Y$ is the collection of ordered pairs whose first component lies in X while second in Y. That is

$$X \times Y = \{(x,y) : x \in X, y \in Y\}.$$

Claim: $X \times Y$ is a set.

Proof: Fix $x \in X$. The assignment $y \to (x,y)$ on index set Y is single valued. Then $\{z : z = (x,y) \text{ for some } y \in Y\}$ is a set.

Thus $Q_x = \{(x,y) : y \in Y\}$ is a set.

Then

$F = \{Q_x : x \in X\}$ is set from axiom 5.

Then $\bigcup F$ is a set.

$X \times Y = \bigcup F$ is a set.

→ One can generalize to

$$X_1 \times X_2 \cdots \times X_n \equiv \prod_{i=1}^{n} X_i$$

$$X^n = \prod_{i=1}^{n} X$$

Collection of n-tuples $(x_1, \ldots, x_n)$.