§ Lecture 5.0

Thursday, 27 August 2026 17:56

Claim: Given a set A, $A \notin A$.

Proof: Construct another set

$$B = \{A\}$$

Assume for contradiction that $A \in A$.

Then $A \in A$ and $A \in B \Rightarrow A \in A \cap B$

$$\Rightarrow A \cap B \neq \phi.$$

Now from axiom 9, either A is not a set or $A \cap B = \phi$. But since A is a set, we must have

$$A \cap B = \phi \quad [\text{Contradiction}]$$

$\Rightarrow$ For any set A, $A \notin A$.

Functions :-

Let X, Y be two sets. Let $P(x, y)$ be a property of $x \in X$ and $y \in Y$ such that for each $x \in X$, there is exactly one $y \in Y$ for which $P(x, y)$ is true.

Then $f: X \to Y$ is defined by P, which for every $x \in X$, assigns $f(x) \in Y$ s.t. $P(x, f(x))$ is true. That is

$$y = f(x) \Leftrightarrow P(x, y) \text{ is true.}$$

X = Domain

Y = Range

Examples: Let $X = N$, $Y = N$

① $P_1(x, y) =$ " $y = x_{++}$ "

Then for each x, $f(x) = x_{++}$ [Explicit]

② $P_2(x, y) =$ " $y_{++} = x$ "

This doesn't define a function on N.

as for $x = 0$, there is no y s.t. $y_{++} = 0$.

But $P_2(x, y)$ defines a function $N \setminus \{0\}$.

by property that on x

$$(f(x))_{++} = x. \quad [\text{Implicit}]$$

③ Constant function: $X = N, Y = N$

$P_3(x, y) =$ " $y = 7$ for any $x \in N$ "

Then $f(x) = 7 \quad \forall x \in N$

Equality of functions: $f: X \to Y$ and $g: X \to Y$ are said to equal, i.e. $f = g$, iff $\forall x \in X \quad f(x) = g(x)$.

Composition of two functions:

Let $f: X \to Y$ and $g: Y \to Z$ be two functions.

Then composition $g \circ f: X \to Z$ is defined as

$$(g \circ f)(x) = g(f(x)) \quad \forall x \in X.$$

[It is important that range of f and domain of g are same otherwise $g \circ f$ is not defined.]

One-to-one functions: A function $f: X \to Y$ is said to be one-to-one or injective iff

$$x \neq x' \Rightarrow f(x) \neq f(x') \quad \forall x, x' \in X.$$

$$\Updownarrow$$

$$f(x) = f(x') \Rightarrow x = x' \quad \forall x, x' \in X$$

Onto / surjective functions:

A function $f: X \to Y$ is said to be onto or surjective iff

$$f(X) = Y \Leftrightarrow \forall y \in Y, \exists x \in X \text{ s.t. } f(x) = y.$$

Ex. $f: N \to N$, defined explicitly as

$$f(n) = n^2 \quad \forall n \in N.$$

Is it is one-to-one:

Attempt: Claim: $n < m \Rightarrow n^2 < m^2$.

Proof: $n < m \Rightarrow n + k = m$ for some $k \in N$ and $k \neq 0$.

$$\begin{aligned} m^2 &= (n+k)^2 \\ &= (n+k)(n+k) \\ &= (n+k)n + (n+k)k \\ &= n^2 + kn + nk + k^2 \\ &= n^2 + (2nk + k^2) \\ &= n^2 + d \end{aligned}$$

Claim: $d \neq 0$. If $d = 0$ then $2nk + k^2 = 0$

$\Rightarrow kk = 0 \Rightarrow k = 0$.

$\Rightarrow m^2 < n^2$.

Now suppose $m^2 = n^2$. For contradiction assume $m \neq n$. Then either $m < n$ or $m > n$.

If $n < m \Rightarrow n^2 < m^2$ contradiction.

If $m < n \Rightarrow m^2 < n^2$ "

$\Rightarrow m = n$.

Is $f(n) = n^2$ onto ?

No because For $y = 2$, there is no $n \in N$ s.t. $n^2 = 2$.

so $f: N \to N$ is not onto.

Suppose $A = \{0, 1, 4, 9, 16, \ldots\}$

Then $f: N \to A$ defined by $f(n) = n^2$ is onto.

It is also one-to-one.

Bijective functions: A function that is both injective and surjective.

Alot depends on domain and range.

§ Lecture 5.1

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Definition (Image of a set): Let $f : X \to Y$ be a function and let $S \subseteq X$. Then define

$$f(S) = \{ f(x) : x \in S \}$$

[Well defined set from axiom 6]

$f(S) =$ (Forward) image of S under f.

Definition (Inverse image):

Let $f : X \to Y$ be a function. Let $W \subseteq Y$. Then

$$f^{-1}(W) = \{ x \in X : f(x) \in W \}$$

$f^{-1}(W) =$ Inverse image of W under f.

Examples: $f : \mathbb{N} \to \mathbb{N}$ $\quad f(n) = 2n+1$

Let $S = \{1, 5, 6\}$

The image of S

$$f(S) = \{ f(1), f(5), f(6) \}$$
$$= \{3, 11, 13\}$$

The inverse image of S:

$$f^{-1}(S) = \{ n \in \mathbb{N} : f(n) \in S \}$$

n	$f(n)$	
0	1	$\in S$
1	3	
2	5	$\in S$
3	7	

$$= \{0, 2\}$$

Composition

$$(f \circ f^{-1})(S) = f(f^{-1}(S))$$
$$= f(\{0, 2\})$$
$$= \{1, 5\} \neq S$$

$$(f^{-1} \circ f)(S) = f^{-1}(f(S))$$
$$= f^{-1}(\{3, 11, 13\})$$
$$= \{1, 5, 6\} = S.$$

i.e. $f^{-1} \circ f (S) = S$ [One-to-one]

$f \circ f^{-1} (S) \neq S$ [Not onto]

Informal: $f : \mathbb{Z} \to \mathbb{Z}$ $\quad f(x) = x^2$.

$$S = \{-1, 0, 1\}$$
$$f(S) = \{ f(x) : x \in S \}$$
$$= \{0, 1\}$$
$$f^{-1} \circ f(S) = f^{-1}(\{0, 1\})$$
$$= \{ x \in \mathbb{Z} : f(x) \in \{0, 1\} \}$$
$$= \{0, 1\} \neq S$$

[Not one to one]
$((-1)^2 = 1^2 \not\Rightarrow 1 = -1.)$

Claim: For a bijective function $f : X \to X$.

$$f \circ f^{-1}(S) = S = f^{-1} \circ f(S). \quad \forall\, S \subseteq X.$$

Proof:

Recall for $f : X \to X$ and $B \subseteq X$

$$f^{-1}(B) = \{ x \in X : f(x) \in B \}$$

Since $f(S)$ is a subset of X. Take

$B = f(S)$

$\Rightarrow$ $f^{-1} \circ f(S) = \{ x \in X : f(x) \in f(S) \}$.

Now $S \subseteq f^{-1} \circ f(S)$:

Let $x \in S$. Then $f(x) \in f(S)$.

$\Rightarrow x \in f^{-1} \circ f(S)$.

$\Rightarrow S \subseteq f^{-1} \circ f(S)$. [Free]

$\Leftarrow$ $f^{-1} \circ f(S) \subseteq S$.

Let $x \in f^{-1} \circ f(S)$

$\Rightarrow f(x) \in f(S)$

$\Rightarrow \exists\, y \in S$ s.t. $f(x) = f(y)$

$\Rightarrow x = y$ (As f is injective)

$\Rightarrow$ $f^{-1} \circ f(S) \subseteq S$

Together $f^{-1} \circ f(S) = S$ [This used only injectivity].

Part2: $f \circ f^{-1}(S) = S$.

$\Leftarrow$ $S \subseteq f \circ f^{-1}(S)$.

Let $y \in S$. Since f is surjective there is some $x \in X$ with $f(x) = y$. That is $f(x) \in S$.

$\Rightarrow x \in f^{-1}(S)$

$\Rightarrow f(x) \in f \circ f^{-1}(S)$

or $y \in f \circ f^{-1}(x)$

$\Rightarrow S \subseteq f \circ f^{-1}(S)$.

$\Leftarrow$ $f \circ f^{-1}(S) \subseteq S$.

Let $y \in f \circ f^{-1}(S)$

Then $y = f(x)$ for some $x \in f^{-1}(S)$.

But $x \in f^{-1}(S) \Rightarrow f(x) \in S$

$\Rightarrow y \in S$

$\Rightarrow$ $\boxed{f \circ f^{-1}(S) \subseteq S.}$ Free.

Together $\boxed{f \circ f^{-1}(S) = S}$

§ Lecture 5.2

Friday, 28 August 2026 11:15

Note that functions are not sets. But we would like sets of functions. That is we want to treat functions as objects.

<u>Axiom 10</u>: Let X and Y be two sets. Then

$$Y^X := \{ f : f : X \to Y \text{ is a function} \}$$

is a set.

<u>Lemma</u>: Let X be a set. Then

$$T = \{ Y : Y \subseteq X \} \text{ is a set.}$$