§ Lecture 4.0

Tuesday, 18 August 2026 12:53

Now we build the objects for sets!

Axiom 2 (Empty set):

There exists a set, called an empty set, denoted as ϕ, which contain no elements. That is for every object x, $x \notin \phi$.

→ There can only be one empty set.

Suppose there are two sets ϕ and ϕ' that are empty. Then

$x \in \phi \iff x \in \phi'$ For all objects x.

This is vacuously true.

$\Rightarrow \phi = \phi'$

Non-empty set: A set that is not empty.

Claim: Let A be a non empty set. Then there exists an object x s.t. $x \in A$.

Proof: Proof by contradiction.

Let there be no object x s.t. $x \in A$.

That is $\forall x$, $x \in A \iff x \in \phi$

Both are equally False statements.

$\Rightarrow A = \phi$ [Contradiction]

Thus there exists an object x s.t. $x \in A$.

Axiom 3 (Singleton sets): If a is an object, then there exists a set $\{a\}$ whose only element is a. The set $\{a\}$ is called the singleton set.

→ **Remark:** For a given object a, the singleton set is unique.

Suppose A and B both are singleton sets for object a.

Then $y \in A \Rightarrow y = a \Rightarrow y \in B \quad \forall y$.

Similarly, $y \in B \Rightarrow y = a \Rightarrow y \in A \quad \forall y$

$\Rightarrow A = B$.

→ The sets

$$\phi \xrightarrow{\text{Axiom } 1,2,3} \{\phi\} \longrightarrow \{\{\phi\}\}$$

singleton set

empty set

Non empty sets

$\phi \neq \{\phi\} \neq \{\{\phi\}\}$.

Axiom 4 (Pairwise union): Let A and B be two sets. Then there exists a set $A \cup B$, called the union of A and B, which is defined as

$$x \in A \cup B \iff (x \in A \text{ or } x \in B)$$

[Elements of $A \cup B$ are elements that are either in A or B or both.

Remark: If $A = A' \Rightarrow A \cup B = A' \cup B$.

Properties:

1. $A \cup B = B \cup A$
2. $(A \cup B) \cup C = A \cup (B \cup C)$
3. $A \cup A = A = A \cup \phi$

Definition (Subsets): Let A and B be sets.

We say that A is subset of B, denoted as $A \subseteq B$ iff

$\forall x \quad x \in A \Rightarrow x \in B$.

We say that A is a proper subset of B, denoted as $A \subset B$ if $A \subseteq B$ and $A \neq B$.

Remark: If $A \subseteq B$ and $A = A'$ then $A' \subseteq B$.

Properties:

1. $A \subseteq A$.
2. $A \subseteq B$ and $B \subseteq A$ then $A = B$.
3. $A \subset B$ and $B \subset C$ then $A \subset C$.

Partial order as it misses the property that given two sets A & B, they might not related by subset relation.

§ Lecture 4.1

Thursday, 20 August 2026 18:34

Axiom 5 (Axiom of specification):-

Let A be a set and let $P(x)$ be a property of $x \in A$. Then there exists a set called

$\{x \in A : P(x) \text{ is true}\}$

whose elements are elements of A for which property $P(x)$ is true.

In other words

$y \in \{x \in A : P(x) \text{ is true}\} \iff (y \in A \text{ and } P(y) \text{ is true})$

Example: $S = \{1, 2, 3, 4, 5\}$

For $x \in S$, $P(x) = "x + 2 = 4"$.

Then $\{x \in S : x + 2 = 4\}$

$= \{2\}$.

- Remark: $\{x \in A : P(x) \text{ is true}\} \subseteq A$.

Definition (Intersection):

$S_1 \cap S_2 = \{x \in S_1 : x \in S_2\}$

That is $x \in S_1 \cap S_2 \Rightarrow x \in S_1$ and $x \in S_2$.

Definition (Disjoint & Distinct):

Two sets A and B are called disjoint if $A \cap B = \phi$.

Two sets A and B are called distinct if $A \neq B$.

Definition (Difference sets):

Given two sets A and B, we define the difference set

$A - B = A \setminus B := \{x \in A : x \notin B\}$.

Claim (Boolean algebra): Let A, B, C be three sets and let X be a set containing A, B, C as subsets. $(A, B, C \subseteq X)$.

1. $A \cup \phi = A$ and $A \cap \phi = \phi$.
2. $A \cup X = X$ and $A \cap X = A$.
3. $A \cup A = A$ and $A \cap A = A$.
4. $A \cup B = B \cup A$ and $A \cap B = B \cap A$.
5. $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$.
6. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

   $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
7. $A \cup (X \setminus A) = X$ and $A \cap (X \setminus A) = \phi$
8. $X \setminus (A \cup B) = (X \setminus A) \cap (X \setminus B)$

   $X \setminus (A \cap B) = (X \setminus A) \cup (X \setminus B)$.

$\Rightarrow$ Transformation / Functions on sets.

Axiom 6 (Replacement):

Let A be a set. For any object $x \in A$ and any object y, let $P(x, y)$ be a property s.t.

"For each $x \in A$, there is at most one y for which $P(x, y)$ is true."

Then there exists a set

$\{y : P(x, y) \text{ is true for some } x \in A\}$.

That is

$z \in \{y : P(x, y) \text{ is true for some } x \in A\}$

$\iff P(x, z)$ is true for some $x \in A$.

Examples:

$A = \{1, 5, 7\}$

Let $P(x, y)$ be property that $y = x{+}{+}$.

So for each x there is exactly one y for which $P(x, y)$ is true.

Then this set exists

$\{y : y = x{+}{+} \text{ for some } x \in A\}$

$= \{2, 6, 8\}$ as $2 = 1{+}{+}$
$6 = 5{+}{+}$
$8 = 7{+}{+}$

In general:

$\{y : y = f(x) \text{ for some } x \in A\}$

$= \{f(x) : x \in A\}$

§ Lecture 4.2

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Axiom 7 (Infinity):

There exists a set N, whose elements are called natural numbers as well as

(i) an object 0 in N

(ii) an object $n{+}{+}$ assigned to each n.

s.t. the Peano axioms hold.

!! We have seen axioms that allow us to build sets. Could we unify them in some way? !!

Axiom 8 (Universal specification):

Suppose for every x we have property $P(x)$. Then there exists a set

$\{x : P(x) \text{ is true}\}$ s.t.

$\forall\, y \in \{x : P(x) \text{ is true}\} \iff P(y)$ is true.

$\longrightarrow$ Every property corresponds to a set.

$\Rightarrow$ This axiom implies most of the previous axioms.

Axiom 2 (Empty set) "$P_2(x) := x \neq x$", which is false for every object.

$\phi = \{x : P_2(x) \text{ is true}\}$.

Axiom 3. (Singleton) Given an object a take

$P_3(x) := $ "$x = a$"

Then $\{a\} = \{x : P_3(x) \text{ is true}\}$.

Axiom 4 (pairwise union): Given set A and B, take $P_4(x) := x \in A$ or $x \in B$. Then

$A \cup B = \{x : P_4(x) \text{ is true}\}$.

Axiom 5 (Specification): Given a set A and a property $Q(x)$ of $x \in A$

$P_5(x) = $ "$x \in A$ and $Q(x)$ is true"

Then $y \in \{x : P_5(x) \text{ is true}\} \iff (y \in A \text{ and } Q(y) \text{ is true})$.

Axiom 6 (Replacement): Given A and $P(x,y)$

$Q(z) = $ "$P(x,z)$ is true for some $x \in A$"

Then $t \in \{z : Q(z) \text{ is true}\} \iff P(x,t)$ is true for some $x \in A$.

!! Axioms 1 & 7 cannot be naturally derived !!

But the problem is much bigger !!

Russell's paradox :-

(1) $P_1(x) = $ "x is a set and $x \notin x$"

$\Omega = \{x : P_1(x) \text{ is true}\}$

$= \{x : x \text{ is a set and } x \notin x\}$

(Valid set from axiom 8)

e.g. $x = \{1,2\}$, then x is a set and $x \notin x$

$\Rightarrow P_1(x)$ is true.

(2) $P_2(x) = $ "x is a set".

$S = \{x : x \text{ is a set}\}$ (Axiom 8)

Here $S \in S$ because S is a set itself.

$\Rightarrow P(S)$ is false.

Question:

Given $\Omega = \{x : x \text{ is a set } \& \ x \notin x\}$, is $\Omega \in \Omega$.

Case 1: Assume $\Omega \in \Omega$. Thus Ω is a set and $\Omega \notin \Omega$.

Case 2: Assume $\Omega \notin \Omega$. Since Ω is already a set so for $\Omega \notin \Omega$, Ω must belong to Ω.

Thus in both cases $\Omega \in \Omega$ and $\Omega \notin \Omega$!!

This is absurd.

§ Lecture 4.3

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How to avoid such paradoxes?

The problem stemmed from a "notion of universal set".

We need some hierarchy -

- $\longrightarrow$ Objects $\rightarrow$ that are not sets
- $\rightarrow$ sets of such objects
- $\longrightarrow$ sets containing the above two

$\vdots$

__Axiom 9 (Regularity):__

If A is a nonempty set, then there is at least one element x of A which is either not a set or is disjoint from A.

So what do we gain?

$\Rightarrow$ Let A be a set. And let $B = \{A\}$.

Then $A \notin A$.

__Proof:__ For contradiction assume $A \in A$.

Since $A \in B$ and $A \in A \Rightarrow A \in A \cap B$

i.e. $A \cap B \neq \phi$. $\longrightarrow$ 1

From axiom 9, either A is not a set or A is disjoint from B.

Since A is a set, we have $A \cap B = \phi$ $\longrightarrow$ 2

contradiction $A \notin A$.

$\Rightarrow$ $A \notin A$ is true for every set.

$\Rightarrow$ $\Omega = \{x : x \text{ is a set and } x \notin x\}$

$S = \{x : x \text{ is a set}\}$

Since $x \notin x$ for all sets. $\Rightarrow \Omega = S$.