§ Lecture 3.0

Monday, 17 August 2026 19:19

Order for natural numbers:

For any two natural numbers a, b, exactly one of the following statement is true.

(i) $a > b$ (ii) $a < b$ (iii) $a = b$.

Proof:

Step 1: Two statements cannot hold simultaneously.

(1a) Let $a > b$ and $a = b$.

$a > b \Rightarrow a \geq b$ and $a \neq b$ [Contradiction]

(1b) Let $a < b$ and $a = b$.

$a < b \Rightarrow a \leq b$ and $a \neq b$ [Contradiction]

(1c) Let $a < b$ and $a > b$

$a < b \Rightarrow a \leq b$ & $a \neq b$

$a > b \Rightarrow a \geq b$ & $a \neq b$

Then $a = b$ & $a \neq b$ contradiction.

(2) One of the statements is true.

Let b is fixed and apply induction on a.

(2a) $a = 0$

Note that $b \geq 0$ $\forall b. \Rightarrow b \geq a$

or either $b > a$ or $b = a$.

(2b) Now we suppose for some a, one of the three statements hold.

(2b).1 $a > b$; then

$a{++} > b$ Done. (One statement is true)

2. $a < b$, then

$a{++} \leq b \Rightarrow a{++} = b$ or $a{++} < b$.

3. $a = b \Rightarrow a{++} > b$ (Exactly one case.)

From induction only one of the statements is true for all a, b.

Note that: We started induction at a base case at 0. What if the base case doesn't start with zero.

Proposition (Strong principle of induction):

Let m_0 be a natural number, and let $P(m)$ be the property of an arbitrary natural number. Assume that for each $m \geq m_0$:

if $P(m')$ is true for all $m_0 \leq m' < m$

then $P(m)$ is also true.

Then $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof:

Define the property

$Q(n) =$ " $P(m')$ holds for all m' with $m_0 \leq m < n$."

Induct on n.

$Q(0) =$ " $P(m')$ holds for all m' with $m_0 \leq m' < 0$."

Since $m' \geq 0 \Rightarrow$ there is no m' s.t. $m' < 0$.

$\Rightarrow$ $P(m')$ holds for all $m_0 \leq m' < 0$.

$\Rightarrow$ $Q(0)$ is true.

Induction step:

Assume that $Q(n)$ holds, ie.

$P(m')$ holds for all m' with $m_0 \leq m' < n$.

Then from hypothesis $P(n)$ is true.

Now we need to show that $Q(n{++})$ holds.

That is " $P(m')$ holds for all m' with $m_0 \leq m' < n{++}$"

Note that

$$m' < n{++} \Rightarrow m'{++} \leq n{++} \Rightarrow n{++} = m'{++} + d$$

$$\Rightarrow n{++} = (m' + d){++} \Rightarrow n = m' + d$$

$$\Rightarrow m' \leq n.$$

So we need to prove that

(1) $P(m')$ is true for all m' with $m_0 \leq m' < n$.

This is true by induction hypothesis.

(2) $P(m')$ is true for all m' with $m_0 \leq m' = n$

This follows from the premise.

$\Rightarrow$ $Q(n{++})$ holds.

$\Rightarrow$ $Q(n)$ is true for all n.

$\Rightarrow$ $P(m)$ is true for all $m_0 \leq m$.

§ Lecture 3.1

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Multiplication:

Let m be a natural number. To multiply zero we define $0 \times m := 0$. Now suppose $n \times m$ is defined recursively, then $n_{++} \times m = n \times m + m$.

Claims.

① $m \times n = n \times m$

Step 1: $m \times 0 = 0$

Base case: $m = 0$

$0 \times 0 = 0$.

Assume: $m \times 0 = 0$

Then $m_{++} \times 0 = m \times 0 + 0 = 0$

Step 2: $n \times m_{++} = n \times m + n$

Base case: $n = 0$

LHS $= 0 \times m_{++} = 0$

RHS $= 0 \times m + 0 = 0$

Assume that $n \times m_{++} = n \times m + n$

then $n_{++} \times m_{++} = n \times m_{++} + m_{++}$

$= n \times m + n + m_{++}$

$= n \times m + m + n_{++}$

$= n_{++} \times m + n_{++}$

$\Rightarrow n \times m_{++} = n \times m + n$

Step 3: $m \times n = n \times m$

Fix m, induct on n.

Base case: $n = 0$

LHS $= m \times 0 = 0$

RHS $= 0 \times m = 0$.

Assume $m \times n = n \times m$

see $m \times n_{++} = m \times n + m$

$= n \times m + m$

$= n_{++} \times m$

$\Rightarrow m \times n = n \times m \quad \forall\, m, n$.

Other properties:

(i) $n \times m = 0 \Leftrightarrow$ At least one of n, m is equal to zero.

If one of n, m is zero then by definition of multiplication $n \times m = 0$. as $0 \times m = n \times 0 = 0 \times 0 = 0$.

$\Rightarrow$ Assume $n \times m = 0 \Rightarrow$ At least one of n, m is zero.

Proof:

Assume m, n are both nonzero. Then we need to show that $n \times m \neq 0$.

Since n is positive $\Rightarrow n = a_{++}$

$n \times m = a_{++} \times m = a \times m + m$

m is a positive number

$\Rightarrow a \times m + m$ is also positive

* $a(b+c) = ab + ac$

Keep a, b fixed. Apply induction on c.

Base $c = 0$, LHS $= a(b+0)$

$= ab$.

RHS $= ab + a \times 0 = ab$.

Assume $a(b+c) = ab + ac$

Then $a(b + c_{++}) = a \times (b+c)_{++}$

$= a \times (b+c) + a$

$= ab + ac + a$

$= ab + a c_{++}$

* $(a \times b) \times c = a \times (b \times c)$.

* $a < b$ and $c > 0$ then $ac < bc$.

Proof: $a < b \Rightarrow a \leq b$ & $a \neq b$

$\Rightarrow a + d = b$ & $a \neq b$

$\Rightarrow a + d = b$ & $d > 0$.

$\Rightarrow (a+d)c = bc$

$\Rightarrow ac + dc = bc$

$\Rightarrow ac < bc$ as $dc \neq 0$.

* $ac = bc$ & $c \neq 0$ then $a = b$.

Proof: For a and b, three cases are possible.

① $a < b \Rightarrow a + d = b$ for $d > 0$.

$\Rightarrow ac + dc = bc$

$\Rightarrow dc = 0 \Rightarrow c = 0$ or $d = 0$

Contradiction.

② $a > b$ similarly.

$\Rightarrow a = b$

Euclidean algorithm:

Let n be a natural number and let q be a positive number. Then there exist natural numbers m & r s.t. $0 \leq r < q$ and

$$n = mq + r.$$

Proof: Induction on n:

Base case: $n = 0$

$= 0q + 0$

i.e. $m = 0,\ r = 0$ where $0 \leq r < q$.

Induction step: suppose $n = mq + r$.

Now

$n_{++} = (mq + r)_{++}$

$= mq + r_{++}$

We have $0 \leq r < q$

Then $r_{++} \leq q$

If $r_{++} = q$ then take $m' = m+1$, $r' = 0$

and $n_{++} = mq + q = (m+1)q + r'$, where

$0 \leq 0 = r' < q$

If $r_{++} < q$, then we are done. by taking $m' = m$, $r' = r_{++}$.

$n_{++} = m'q + r'$!

This completes the proof.

Exponentiation:

$m^0 = 1$ (m to the power 0)

m^n is defined.

$$m^{n_{++}} = m^n \times m$$

Story:

$a+b$	$a-b$?
$a \times b$	a/b ?
a^b	a^{-b} ?

§ Lecture 3.2

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Set (Informal) :- A set A is any unordered collection of objects.

If x is an object, we say that $x \in A$ if x lies in collection. otherwise $x \notin A$.

This is not good enough. unless we can define some operations on them.

Axiom 1. (Sets are objects themselves).

If A is set, then A is also an object.

If A and B are sets then it is meaningful to ask $A \in B$ or not.

!! But not all objects are sets. For example we don't typically consider a natural number to be set. !!

Def. (Equality of sets)

Two sets A and B are equal, $A = B$, iff every element of A is an element of B and vice-versa.

→ Remark 1: The relation of set equality is

1. Reflexive: $A = A$.
2. Symmetric: $A = B \Rightarrow B = A$
3. Transitive: $A = B$ & $B = C \Rightarrow A = C$

Equivalence relation.

→ Remark 2:

The set equality obeys the axiom of substitution w.r.to relation '$\in$'. That is if $x \in A$ and $A = B$ then $x \in B$.

Which objects are sets??

We shall build them just like we did for natural numbers.

Axiom 2. (Empty set)

There exists a set ϕ known as the empty set, which contains no elements, i.e. for all x, $x \notin \phi$.

!! There can't be two empty sets !!

Let ϕ & ϕ' be two empty sets.

$\Rightarrow \phi \subseteq \phi'$: $\forall x$ if $x \in \phi \Rightarrow x \in \phi'$

[This is true because the condition $x \in \phi$ is false.]

Similarly $\phi' \subseteq \phi \Rightarrow \phi = \phi'$!

"A set that is not empty is called non empty set."

Claim: Let A be a nonempty set then there exists an object x s.t. $x \in A$.

Proof: Proof by contradiction

Let there doesn't exist any x s.t. $x \in A$.

$\Rightarrow \forall x \quad x \notin A \quad \& \quad x \notin \phi$

ie. $\forall x$, $x \in \phi \Leftrightarrow x \in A$.

$\Rightarrow A = \phi$ (contradiction).