§ Lecture 21.0

Monday, 2 November 2026 20:22

Series Laws: let $\sum_{n=m}^{\infty} a_n = x$, $\sum_{n=m}^{\infty} b_n = y$. Then

① $\sum_{n=m}^{\infty} (a_n + b_n) = x + y$

② For any $c \in \mathbb{R}$, $\sum_{n=m}^{\infty} c\,a_n = cx$

③ $\sum_{n=m}^{\infty} a_n$ is convergent iff $\sum_{n=m+k}^{\infty} a_n$ for any $k \geq 0$.

In fact, $\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n$.

[Note that we are not rearranging here.]

Claim: An infinite series $\sum_{n=m}^{\infty} a_n$ of non-negative real numbers is convergent iff there exists a real number M s.t.

$$\sum_{n=m}^{N} a_n \leq M \quad \forall\, N \geq m.$$

Proof: Recall that an increasing sequence is convergent iff it is bounded from above.

Clearly, $(S_N)_{N \geq m}$, where $S_N = \sum_{n=m}^{N} a_n$ is an increasing sequence as $a_n \geq 0 \ \forall\, n \geq m$.

Then $(S_N)_{N \geq m}$ is convergent iff it is bounded.

Lemma (Comparison test). Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be infinite series and $|a_n| \leq b_n \ \forall\, n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent, then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. Further,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Proof: Given: $\sum_{n=m}^{\infty} b_n$ is convergent.

$\Rightarrow (S_N)_{N \geq m}$, where $S_N = \sum_{n=m}^{N} a_n$ is

(i) Increasing as $b_n \geq |a_n| \geq 0$

(ii) Convergent.

This means $(S_N)_{N \geq m}$ is bounded. Let $M \in \mathbb{R}$ s.t.

$$S_N \leq M.$$

Let $T_N = \sum_{n=m}^{N} |a_n|$. Then $T_N = \sum_{n=m}^{N} |a_n| \leq \sum_{n=m}^{N} b_n \leq M$.

$\Rightarrow (T_N)_{N \geq m}$ is an increasing sequence, that is bounded. Then $(T_N)_{N \geq m}$ is convergent, thus $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Since $T_N \leq S_N \Rightarrow \lim_{N \to \infty} T_N \leq \lim_{n \to \infty} S_N$

$$\Rightarrow \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

We already have proven $\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|$ for absolutely convergent series.

Example:

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \text{Divergent if } |x| \geq 1 \\ \text{Absolutely convergent if } |x| < 1. \end{cases}$$

and $= \frac{1}{1-x}$.

Solution: Let $a_n = x^n$. Then $|a_n| = |x|^n \geq 1. \ \forall\, n \geq 0$.

Thus for $\varepsilon = \frac{1}{2}$, there is no $N \geq m$ s.t. $|a_n| \leq \frac{1}{2} \ \forall\, n \geq N$.

(Because $|a_n| \geq 1 \ \forall\, n \geq m$).

$\Rightarrow \lim_{n \to \infty} a_n \neq 0$

$\Rightarrow \sum_{n=0}^{\infty} x^n$ is not convergent when $|x| \geq 1$.

Case 2: $|x| < 1$,

$$S_N = \sum_{n=0}^{N} x^n = \frac{1 - x^{N+1}}{1 - x}.$$

$$x S_N = \sum_{n=0}^{N} x^{n+1} = \sum_{n=1}^{N+1} x^n = -1 + S_N + x^{N+1}$$

$$\Rightarrow (1 - x) S_N = 1 - x^{N+1}$$

$$\text{or} \quad S_N = \frac{1 - x^{N+1}}{1 - x}$$

$$= \frac{1}{1-x} - \frac{1}{1-x} x^{N+1}$$

For $|x| < 1$, $\lim_{N \to \infty} x^N = 0 = \lim_{N \to \infty} x^{N+1}$

$$\Rightarrow \lim_{N \to \infty} S_N = 1/1-x$$

$$\text{or} \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \forall\, |x| < 1.$$

Cauchy criterion: Let $(a_n)_{n \geq 1}$ be a sequence where $a_n \geq 0$ and $a_{n+1} \leq a_n \ \forall\, n \geq 1$. Then series $\sum_{n=1}^{\infty} a_n$ is convergent iff $\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + \cdots$ is convergent.

Ex.

$$\sum_{n=1}^{\infty} \frac{1}{n^q} = \begin{cases} \text{Divergent if } 0 < q \leq 1 \\ \text{Convergent if } q > 1. \end{cases}$$

Solution. Note that $\frac{1}{n^q} \geq 0 \quad \forall\, q > 0$.

Further $(n+1) > n \qquad (n \geq 1)$

$$\Rightarrow (n+1)^q > n^q \Rightarrow \frac{1}{(n+1)^q} < \frac{1}{n^q} \quad \text{(Decreasing)}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent iff $\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^q}$ is convergent.

Let $x = 2^{(1-q)}$, then we need to find q s.t. $\sum_{k=0}^{\infty} x^k$ is convergent.

We have proven that it is convergent for $|x| < 1$.

i.e. $|2^{1-q}| < 1. \iff 2^{1-q} < 1 \quad \text{As } 2^{1-q} \geq 0.$

$$\iff 1 - q < 0$$

$$\iff q > 1.$$

§ Lecture 21.1

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Rearrangement of series:

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{k} a_n + \sum_{n=k+1}^{\infty} a_n \quad \text{[Not a rearrangement.]}$$

$$\downarrow$$

$$\underbrace{\sum_{n=m}^{\infty} a_{f(n)}}_{\text{Rearrangement of } \sum_{n=m}^{\infty} a_n}, \text{ where } f \text{ is a bijective map from } \mathbb{N} \to \mathbb{N}.$$

Grouping: Consider a series $\sum_{n=0}^{\infty} a_n$. Choose indices

$n_0 < n_1 < n_2 \cdots$ and write

$$b_0 = \sum_{n=0}^{n_0} a_n, \quad b_1 = \sum_{n=n_0+1}^{n_1}, \cdots$$

or $$b_k = \sum_{n=n_{k-1}+1}^{n_k} a_n \quad (k \geq 1)$$

Then the grouped series $\sum_{k=0}^{\infty} b_k$.

e.g. $\sum_{n=0}^{\infty} (-1)^n$

Choose $n_k = 2k+1$, $n_0 = 1$, $n_1 = 3, \cdots$

$$b_0 = \sum_{n=0}^{n_0} a_n = \sum_{n=0}^{1} a_n = 0$$

$$b_1 = \sum_{n=2}^{3} a_n = 0$$

$$\vdots$$

$$\sum_{n=0}^{\infty} (-1)^n = \overbrace{(1-1)}^{b_0} + \overbrace{(1-1)}^{b_1} + \cdots$$

[Not rearrangement.]

Now let us consider the partial sums:

$$S_N = \sum_{n=0}^{N} a_n, \qquad T_N = \sum_{n=0}^{N} b_n.$$

$(S_n)_{n\geq 0}$ & $(T_k)_{k\geq 0}$. ; $T_0 =$

Notice $S_{n_0} = T_0$

$S_{n_1} = T_1$

$\vdots$

$S_{n_k} = T_k$

That is $(T_k)_{k\geq 0}$ is a subsequence of sequence $(S_n)_{n\geq 0}$.

All the convergence properties of $\sum_{n=0}^{\infty} b_n$ can be obtained by by analyzing the subsequence of the sequence of partial sums of $\sum_{n=0}^{\infty} a_n$.

Clearly, if $\sum_{n=0}^{\infty} a_n$ is convergent, then $\sum_{n=0}^{\infty} b_n$ is convergent for any regrouping.

While a nonconvergent sequence may give rise to a convergent subsequence as happened above.

In the above sense rearrangement is special.

Theorem: Let $\sum_{n=m}^{\infty} a_n$ be an absolutely convergent series.

and let $f: \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=m}^{\infty} a_{f(n)}$ is also absolutely convergent and in fact

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{\infty} a_{f(n)}.$$

Example: Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

We have shown that $\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is not absolutely convergent. While it is convergent (from alternating series test)

Let us analyze the following:

	Series	How is it obtained
A	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$	original
G	$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots$	Grouping of A
A'	$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots$	Rearranging
G'	$\left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \cdots$	Grouping A'

$\downarrow$

n^{th} block $\quad \frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}$

Let S_N be the partial sums of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, i.e.

$$S_N = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n}$$

$$S_{2n} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k}\right) \quad \text{(Every term is positive and } s_2 = \tfrac{1}{2}\text{)}$$

$\Rightarrow S_{2n} \geq \frac{1}{2} \quad \forall n.$

$\Rightarrow (S_{2n})_{n\geq 1}$ converge to where series converges. and it is $\geq \frac{1}{2}$.

Rearranged case:

$$\frac{1}{2n-1} - \frac{1}{4n-2} - \frac{1}{4n}$$

$$= \frac{1}{4n-2} - \frac{1}{4n}$$

$$= \frac{1}{2}\left[\frac{1}{2n-1} - \frac{1}{2n}\right]$$

Skipping the first n blocks

Let S'_N be partial sum for A'

$$S'_{3n} = \frac{1}{2} S_{2n} \qquad n \geq 1$$

$$S'_{3n+1} = S'_{3n} + \frac{1}{2n+1}, \quad S'_{3n+2} = S'_{3n} + \frac{1}{4n+1} - \frac{1}{4n+2}$$

All partial sums of A'

If A converges to L, i.e. $S_{2n} \to L$

$\Rightarrow$ A' converges to $\frac{1}{2}L < L$

Thus rearrangement altered the sum.

Riemann rearrangement phenomenon:

Any convergent series which is not absolutely convergent can be rearranged to converge to any value.

§ Lecture 21.2

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Theorem [The root test]

Let $\sum_{n=m}^{\infty} a_n$ be an infinite series. Let

$\alpha = \limsup_{n\to\infty} |a_n|^{y_n}$. Then

if ① $\alpha < 1$ $\sum_{n=m}^{\infty} a_n$ is absolutely convergent.

② $\alpha > 1$ " " divergent.

③ $\alpha = 1$ Inconclusive.

Theorem [The ratio test]: Let $\sum_{n=m}^{\infty} a_n$ be an infinite series with $a_n \neq 0 \ \forall n \geq m$. Let $\beta = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. Then if

① $\beta < 1$ series is absolutely convergent

② $\beta > 1$ Divergent

③ $\beta = 1$ Inconclusive.

The ratio test follows from the following lemma.

Let $(a_n)_{n\geq m}$ be sequence of positive reals. Then

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} a_n^{1/n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

Proof of root test: