§ Lecture 20.0

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Infinite series:

Def An infinite series is an expression of the form

$$\sum_{n=m}^{\infty} a_n \equiv a_m + a_{m+1} + \cdots ,$$

where m is an integer and $a_n \in \mathbb{R}$ $\forall\, n \geq m$.

Convergence of series:

Let $\sum_{n=m}^{\infty} a_n$ be a formal series and let:

$$S_N := \sum_{n=m}^{N} a_n \ , \ N \geq m \qquad \left(N^{th} \text{ partial sum}\right)$$
$$\left[S_N := 0 \text{ if } N < m\right]$$

If $\lim_{N\to\infty} S_N = L$, then we say that the infinite series is convergent and converges to L. We then write

$$L = \sum_{n=m}^{\infty} a_n \qquad \hookrightarrow L \text{ is unique.}$$

If sequence $(a_n)_{n\geq m}$ doesn't converge, we say that the series diverges and we denote using one number to $\sum_{n=m}^{\infty} a_n$.

Claim 1: Let $\sum_{n=m}^{\infty} a_n$ be an infinite series of reals Then

$\sum_{n=m}^{\infty} a_n$ converges iff $\forall\, \varepsilon > 0$ $\exists$ $N \geq m$ s.t

$$\left|\sum_{n=p}^{q} a_n\right| < \varepsilon \qquad \forall \quad q \geq p \geq N.$$

Proof: ($\Rightarrow$) Let $\sum_{n=m}^{\infty} a_n$ be convergent Then $(S_N)_{N\geq m}$, where

$$S_N = \sum_{n=m}^{N} a_n \text{ is convergent and hence a Cauchy sequence.}$$

That is $\forall\, \varepsilon > 0$ $\exists\, N \geq m$ s.t:

$$|S_q - S_p| < \varepsilon \qquad \forall \quad q \geq p \geq N.$$

i.e. $\left|\sum_{n=m}^{q} a_n - \sum_{n=m}^{p} a_n\right| < \varepsilon$

or $\left|\sum_{n=p}^{q} a_n\right| < \varepsilon$.

Thus $\forall\, q \geq p$, $\sum_{n=p}^{q} a_n = 0 < \varepsilon$ (Trivially)

($\Leftarrow$) Assume there is $N \geq m$ s.t $\forall\, q \geq p \geq N$

$$\left|\sum_{n=p}^{q} a_n\right| < \varepsilon \qquad \forall\, \varepsilon > 0$$

$$\Rightarrow \quad |S_q - S_p| < \varepsilon.$$

$\Rightarrow$ $(S_N)_{N\geq m}$ is Cauchy sequence and hence convergent.

Claim: If $\sum_{n=m}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

Proof: Let $S_N = \sum_{n=m}^{N} a_n$, then $(S_N)_{N\geq m}$ is a convergent sequence.

That is $\forall\, \varepsilon > 0$ $\exists\, N \geq m$ s.t:

$$|S_p - S_q| < \varepsilon \qquad \forall \quad p \geq q \geq N.$$

Choose $q = p+1$

$$\Rightarrow \quad |S_p - S_{p+1}| < \varepsilon$$

or, $|a_{p+1}| < \varepsilon$.

That is $\forall\, \varepsilon > 0$ $\exists\, N \geq m$ s.t:

$$|a_q - 0| < \varepsilon \quad \forall \quad q \geq N \quad (\text{since } q = p+1).$$

$$\Rightarrow \lim_{q\to\infty} a_q = 0$$

Ex: Let $a_n = 1 \quad \forall\, n \geq 1$. Then sequence $(a_n)_{n\geq 1}$ is convergent and $\lim_{n\to\infty} a_n = 1 \neq 0$

$$\Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} 1 \text{ is divergent.}$$

Absolute convergence: $\sum_{n=m}^{\infty} a_n$ is said to be absolutely convergent

iff $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Claim: If a series $\sum_{n=m}^{\infty} a_n$ is absolutely convergent, then it is also convergent. Further, in this case

$$\left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n|$$

Proof: Given $\sum_{n=m}^{\infty} |a_n|$ is convergent. Then

$\forall\, \varepsilon > 0$ $\exists\, N \geq m$ s.t: $\left|\sum_{n=p}^{q} |a_n|\right| < \varepsilon \quad \forall\, q \geq p \geq N$

But $\left|\sum_{n=p}^{q} a_n\right| \leq \sum_{n=p}^{q} |a_n| = \left|\sum_{n=p}^{q} |a_n|\right| < \varepsilon \quad \forall\, q \geq p \geq N$

$\Rightarrow \sum_{n=m}^{\infty} a_n$ is convergent

Let $L_N = \sum_{n=m}^{N} a_n$ $T_N = \sum_{n=m}^{N} |a_n|$

Then $(L_N)_{N\geq m}$ and $(T_N)_{N\geq m}$ both are convergent

Note that: $|L_N| = \left|\sum_{n=m}^{N} a_n\right| \leq \sum_{n=m}^{N} |a_n| = T_N$

$$\Rightarrow \quad -T_N \leq S_N \leq T_N$$

$$\Rightarrow \quad -\lim_{N\to\infty} T_N \leq \lim_{N\to\infty} S_N \leq \lim_{N\to\infty} T_N$$

$$-\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} a_n \leq \sum_{n=m}^{\infty} |a_n|$$

$$\Rightarrow \quad \left|\sum_{n=m}^{\infty} a_n\right| \leq \sum_{n=m}^{\infty} |a_n|$$

Lemma (Alternating series test): Let $(a_n)_{n\geq m}$ be a sequence of real numbers with $a_n \geq 0$ and $a_{n+1} \leq a_n$ $\forall\, n \geq m$. Then $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent iff $\lim_{n\to\infty} a_n = 0$.

Proof: ($\Rightarrow$) If $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent then

$$\lim_{n\to\infty} (-1)^n a_n = 0$$

i.e. $\forall\, \varepsilon > 0$ $\exists\, N \geq m$ s.t: $\left|(-1)^n a_q - 0\right| < \varepsilon \quad \forall\, q \geq N$.

or $|a_n| < \varepsilon \quad \forall\, n \geq N$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

($\Leftarrow$) Suppose $\lim_{n\to\infty} a_n = 0$, our goal is to show that $(S_N)_{N\geq m}$ where $S_N = \sum_{n=m}^{N} (-1)^n a_n$, is a convergent sequence.

Note that:

$$S_{N+2} = S_N + (-1)^{N+1} a_{N+1} + (-1)^{N+2} a_{N+2}$$

$$= S_N + (-1)^{N+2} \underbrace{\left(a_{N+1} - a_{N+2}\right)}_{\geq 0}$$

$$\Rightarrow \quad S_{N+2} \geq S_N \qquad \text{if } N \text{ is odd.}$$

and $S_{N+2} \leq S_N$ if N is even.

Assume N is even, then from inductive:

$$S_{N+2k} \leq S_N \qquad \forall\, k \in \mathbb{N}.$$

Then base case: $k=0$; $S_N = S_N$.

Assume $S_{N+2k} \leq S_N$

Then $S_{N+2k+2} = S_{N+2k} + (-1)^{N+2k+1}\left(a_{N+2k+1} - a_{N+2k+2}\right)$

$$= S_{N+2k} - \left(a_{N+2k+1} - a_{N+2k+2}\right)$$

$$\leq S_{N+2k} \ ]$$

Now $N+1$ is odd. $\Rightarrow$

$$S_{N+1+2} \geq S_{N+1}$$

From inductive $S_{N+1+2k} \geq S_{N+1} = S_N - a_{N+1}$

Further, $S_{N+1+2k} = S_{N+2k} - a_{N+2k+1}$

$$\leq S_{N+2k}.$$

Thus

$$S_N - a_{N+1} \leq S_{N+1+k} \leq S_{N+2k} \leq S_N \qquad \forall\, k \in \mathbb{N}$$

$$\Rightarrow \quad S_N - a_{N+1} \leq S_N \leq S_N \qquad \forall\, n \geq N. \quad \longrightarrow (1)$$

Note that $\lim_{n\to\infty} a_n = 0$

$$\Rightarrow \forall\, \varepsilon > 0 \quad \exists\, n \geq m \quad \text{s.t:} \quad |a_n| < \varepsilon \quad \forall\, k \geq n.$$

Let $N := M$ if M is even or $N = M+1$ when M is odd.

$\Rightarrow N$ is even, $N+1 > M$

$$\Rightarrow \quad \forall\, \varepsilon > 0 \quad |a_{N+1}| < \varepsilon. \quad \text{or} \quad a_{N+1} < \varepsilon \ \text{ as } a_{N+1} \geq 0.$$

Let $j, k \geq N$ and since $|S_j - S_k| = |S_k - S_j|$ assume that:

$$S_j \geq S_k.$$

Adding (1) to get

$$S_j \leq S_N \ , \ S_k \geq S_N - a_{N+1}$$

$$\Rightarrow \quad 0 \leq S_j - S_k \leq S_N - S_N + a_{N+1}$$
$$= a_{N+1} < \varepsilon.$$

$$\Rightarrow \quad |S_j - S_k| < \varepsilon \quad \forall \quad j, k \geq N \text{ and } \forall\, \varepsilon > 0$$

$\Rightarrow (S_j)_{j\geq m}$ is a Cauchy sequence and hence convergent.

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Claim: Let $\sum_{n=m}^{\infty} a_n$ be an infinite series of reals with $a_n \geq 0 \; \forall \, n \geq m$.

Then the series is convergent iff $\exists$ a real number M s.t.

$$S_N = \sum_{n=m}^{N} a_n \leq M \quad \forall \text{ integers } N \geq m.$$

Proof: Recall that a sequence $(S_N)_{N \geq m}$ of positive reals is convergent iff it is increasing and upper bounded.

Clearly $S_{N+1} \geq S_N$ as $a_n \geq 0 \; \forall \, n \geq m$.

Then

$(S_N)_{N \geq m}$ is bounded $\Rightarrow \sum_{n=m}^{\infty} a_n$ is convergent.

Similarly the other direction.

Comparison test: Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two infinite series.

Suppose $|a_n| \leq b_n \; \forall \, n \geq m$. Then if $\sum_{n=m}^{\infty} b_n$ is convergent

then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent. Further,

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$$

Proof: Given $\quad 0 \leq |a_n| \leq b_n \;\Rightarrow\; b_n \geq 0 \quad \forall \, n \geq m.$

let $\quad S_N = \sum_{n=m}^{N} a_n, \quad T_N = \sum_{n=m}^{N} |a_n|, \quad P_N = \sum_{n=m}^{N} b_n.$

Clearly $(T_N)_{N \geq m}$ and $(P_N)_{N \geq m}$ are increasing sequences.

If $\sum_{n=m}^{\infty} b_n$ is convergent $\Rightarrow$ $(P_N)_{N \geq m}$ is convergent and hence a bounded sequence. That is $\exists M > 0$ s.t. $P_N \leq M$.

But $\quad T_N = \sum_{n=m}^{N} |a_n| \leq \sum_{n=m}^{N} b_n = P_N \leq M.$

$\Rightarrow (T_N)_{N \geq m}$ is convergent too.

$\Rightarrow \sum_{n=m}^{\infty} |a_n|$ is convergent. $\Rightarrow \sum_{n=m}^{\infty} a_n$ is convergent too.

$\Rightarrow (S_N)_{N \geq m}$ is convergent.

Clearly $\quad T_N \leq P_N \Rightarrow \lim_{N \to \infty} T_N \leq \lim_{N \to \infty} P_N$

or $\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n.$

Since $\quad |S_N| \leq T_N \quad$ or $\quad -T_N \leq S_N \leq T_N$

$$\Rightarrow \left| \lim_{N \to \infty} S_N \right| \leq \lim_{N \to \infty} T_N$$

$$\text{or } \left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Geometric series:

$$\sum_{n=0}^{\infty} x^n = \begin{cases} \frac{1}{1-x} & \text{when } |x| < 1 \\ \text{divergent when } |x| \geq 1 \end{cases}$$

Sol^n: Case 1. $|x| \geq 1$.

Then $\quad |x^n - 0| = |x^n| = |x|^n \geq 1.$

take $\varepsilon = 1/2$, then there is no $N \geq 0$ s.t.

$$|x^n - 0| < 1/2 \quad \forall \, n \geq N \quad (\text{as } |x^n - 0| \geq 1).$$

$\Rightarrow \lim_{n \to \infty} x^n \neq 0 \Rightarrow (x^n)_{n \geq 0}$ is not a convergent sequence when $|x| \geq 1$.

$\Rightarrow \sum_{n=0}^{\infty} x^n$ is not convergent when $|x| \geq 1$.

Case 2. $|x| < 1$.

Then $\quad S_N = \sum_{x=0}^{N} x^n$

$$x S_N = \sum_{x=0}^{N} x^{n+1} = x + \cdots + x^{N+1}$$

$$= \sum_{x=0}^{N} x^n + x^{N+1} - 1$$

$$= S_N + x^{N+1} - 1$$

or $\quad S_N = \frac{x^{N+1} - 1}{x - 1} = \frac{1 - x^{N+1}}{1 - x}$

$$= \frac{1}{1-x} - \frac{1}{1-x} x^{N+1}$$

$$\lim_{N \to \infty} S_N = \frac{1}{1-x} - \frac{1}{1-x} \lim_{N \to \infty} x^{N+1} \quad (\text{As } (x^n)_{n \geq 0} \text{ is convergent sequence when } |x| < 1)$$

$$\lim_{N \to \infty} S_N = \frac{1}{1-x}$$

or $\quad \boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}}$

Cauchy criterion: Let $(a_n)_{n \geq 1}$ be a decreasing sequence of nonnegative reals. Then $\sum_{n=1}^{\infty} a_n$ is convergent iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k} \equiv a_1 + 2 a_2 + 4 a_4 + \cdots$$

is convergent.

Use case: $\quad \sum_{n=1}^{\infty} \frac{1}{n^q} = \begin{cases} \text{convergent} & \text{if } q > 1, \; q \in \mathbb{Q} \\ \text{divergent} & \text{if } \alpha < q \leq 1, \; q \in \mathbb{Q} \end{cases}$

(Riemann zeta function)

Proof:

$$a_n = n^{-q} \qquad [\text{This is decreasing if } q > 0]$$

$$a_{2^k} = (2^k)^{-q}$$

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \left(2^k\right)^{-q}$$

$$= \sum_{k=0}^{\infty} \left(2^{(1-q)}\right)^k$$

$$= \sum_{k=0}^{\infty} x^k, \quad \text{where } x = 2^{1-q}$$

if $q > 1 \Rightarrow 1 - q < 0 \Rightarrow 2^{1-q} < 2^0 = 1$, i.e. $|x| < 1$.

if $0 < q \leq 1 \Rightarrow 0 \leq 1 - q < 1 \Rightarrow 2^{1-q} \geq 1$, i.e. $|x| \geq 1$.

Thus from Cauchy criterion

$$\sum_{n=1}^{\infty} \frac{1}{n^q} = \begin{cases} \text{convergent} & \text{when } q > 1 \\ \text{divergent} & \text{when } \alpha < q \leq 1. \end{cases}$$