§ Lecture 2.0

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We have 5 Peano axioms.

Assumption:

There exists a number system N, whose elements we will call natural numbers, for which axioms 1-5 are true.

Why is it an assumptions?

⇒ Peano axioms can be checked if a 'number system' already exists. Otherwise you can have all axioms you like, there might not be any realization.

⇒ More importantly, there will eventually be a need to assume something

— Godel's incompleteness theorems.

Let us assume notions of "finite" and "infinite". Then all natural numbers are finite assuming 0 is finite.

Proof: $P(n) = n$ is finite.

$P(0) = 0$ is finite (true by assumption)

Assume $P(n) =$ "n is finite" is true.

Then $P(n{+}{+}) =$ "$n{+}{+}$ is finite" is also true.

⇒ All natural numbers are finite.

However, there are infinitely many natural numbers as the increment op. never stops.

Proposition (Recursive definitions) :-

Let $f_n : N \to N$ be functions for each natural number n. Let c be a natural number. Then for each n we can assign an unambiguous — exactly one — natural number a_n s.t. $a_0 = c$ and $a_{n++} = f(a_n)$ for each n.

Proof: a_n cannot be redefined, however the values of a_n can be same for many n.

Induction on n:

By assumption $a_0 = c$. This cannot be redefined again otherwise it will demand $n{+}{+} = 0$ for some n. [This will violate axiom 3]

Assume a_n is assigned unambiguously. Then a_{n++} is also unambiguous otherwise we will have $n{+}{+} = m{+}{+}$ for some $m \neq n$.

[Violates assumption 4]

⇒ a_n is defined unambiguously for each n.

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Addition : Let m be a natural number. To add zero to it we define $0+m := m$.

Now suppose inductively that we defined adding n to m. Then

$$n_{++} + m := (n+m)_{++}$$

$\Rightarrow$ $1+m = 0_{++} + m = (0+m)_{++} = m_{++}$

$2+m = 1_{++} + m = (1+m)_{++} = (m_{++})_{++}$

$\vdots$

What we didn't define
$m+0 = ?$
$m+n_{++} = ?$
$m+n \stackrel{?}{=} n+m.$

We defined $0+m$ & $n_{++}+m = (n+m)_{++}$

Claim 1: $m+0 = m$ $\forall$ natural numbers m.

Proof: Induction on m:

$\rightarrow$ $m=0$; LHS $= 0+0 = 0$
RHS $= 0$
LHS = RHS.

$\rightarrow$ Assume it is true for m, ie. $m+0 = m$

We need to prove that $m_{++}+0 = m_{++}$.

$m_{++}+0 = (m+0)_{++}$
$= m_{++}$.

$\Rightarrow$ $m+0 = m$ is true for all m.

Claim 2: $m+n_{++} = (m+n)_{++}$.

Proof: Let us do induction on m.

$\Rightarrow$ For $m=0$
LHS $= 0+n_{++} = n_{++}$
RHS $= (0+n)_{++} = n_{++}$
LHS = RHS.

$\Rightarrow$ Assume $m+n_{++} = (m+n)_{++}$

Then we need to show that $m_{++}+n_{++} = (m_{++}+n)_{++}$

$\rightarrow$ $m_{++}+n_{++} = (m+n_{++})_{++}$
$= ((m+n)_{++})_{++}$
$= (m_{++}+n)_{++}$

Thus $m+n_{++} = (m+n)_{++}$ $\forall$ m.

Claim 3 $m+n = n+m$ for all natural numbers.

Proof: Induction on m.

$\Rightarrow$ $m=0$ case
LHS $= 0+n = n$
RHS $= n+0 = n$.
LHS = RHS.

$\Rightarrow$ Assume $m+n = n+m$ for some m.

Then we need to show that $m_{++}+n = n+m_{++}$.

$\rightarrow$ $m_{++}+n = (m+n)_{++}$
$= (n+m)_{++}$
$= n+m_{++}$

Thus $m+n = n+m$ $\forall$ natural numbers. m, n.

No property of n was used. Other than it is a natural number.

Other properties:

1. $(a+b)+c = a+(b+c)$

Apply induction on c keeping a, b fixed.

$\rightarrow$ $c=0$ case : $(a+b) = a+(b+0) = a+b$.

$\rightarrow$ $(a+b)+c_{++} = ((a+b)+c)_{++}$
$= (a+(b+c))_{++}$
$= a+(b+c)_{++}$
$= a+(b+c_{++})$.

2. $a+b = c+b \Rightarrow a=c$.

Definition (Positive natural numbers):

A natural number is said to be positive iff it is not zero.

Claim: If a is positive and b is any natural number than $a+b$ is positive.

[Induction on b].

Claim: $a+b=0 \Rightarrow a=0$ & $b=0$.

Ordering of natural numbers :

Let n, m be natural numbers. We say that n is greater than or equal to m, $n \geq m$ or $m \leq n$ iff $n = m+a$ for some natural number a.

We say that n is strictly greater than m $n > m$ or $m < n$ iff $n \geq m$ & $n \neq m$.

Claims:

① $a \geq a$ $[a = a+0]$

② $a \geq b$ and $b \geq c \Rightarrow a \geq c$.

$a = b+d$; $b = c+e \Rightarrow a = c+d+e$

③ $a \geq b$ & $b \geq a \Rightarrow a=b$

$a = b+d$, $b = a+c$

$a = a+c+d \Rightarrow 0 = c+d \Rightarrow c=0$ & $d=0$

$\Rightarrow a=b$.

④ $a \geq b$ iff $a+c \geq b+c$.

$\Rightarrow$ $a \geq b \Rightarrow a = b+c \Rightarrow a+c = (b+c)+c$
$\Rightarrow a+c \geq b+c$.

$\Leftarrow$ $a+c \geq b+c \Rightarrow a+c = b+c+d \Rightarrow a = b+d$
$\Rightarrow a \geq b$.

⑤ $a < b$ iff $a_{++} \leq b$.

$\Rightarrow$ $a < b \Rightarrow a \leq b$; $a \neq b$
$\Rightarrow a+c = b$, $a \neq b$
$\Rightarrow a+c = b$, $c \neq 0$
$\Rightarrow a+d_{++} = b$
$\Rightarrow a_{++}+d = b$
$\Rightarrow a_{++} \leq b$.

$\Leftarrow$ $a_{++} \leq b \Rightarrow a_{++}+d = b$
$\Rightarrow (a+d)_{++} = b$
$\Rightarrow a+d_{++} = b$
$\Rightarrow a \leq b$
if $a=b$ then
$a+0 = a = a+d_{++}$
$\Rightarrow d_{++} = 0$ Can't be true

So $a \neq b$ & $a \leq b \Rightarrow a < b$.