§ Lecture 19.0

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Recall: for $x \geq 0$ and $n \geq 1$

$$x^{1/n} := \sup E, \quad \text{where } E = \{y \in \mathbb{R} : y \geq 0 \text{ and } y^n \leq x\}.$$

Note: ① Supremum exists.

② The set E is bounded by $\max(1, x)$, i.e. $\forall y \in E,\ y \leq \max(1, x)$

Proof: Suppose $y > \max(1, x)$ for some $y \in E$.

Then $y^n > y$ (As $y > 1$)

$> x$ (As $y > x$)

This is a contradiction

Lemma: Let $x, y \geq 0$ and $n \geq 1$. Then

① $y = x^{1/n}$ iff $y^n = x$.

② $x^{1/n} \geq 0$.

③ $x > y \iff x^{1/n} > y^{1/n}$.

Proof ①: Assume $y = x^{1/n}$.

Because y^n and x are real numbers, either

① $y^n < x$ ② $y^n > x$ ③ $y^n = x$.

We will show that first two cases are impossible.

Case 1: $y^n < x$.

Clearly we could find $\varepsilon > 0$ s.t. $y^n + \varepsilon < x$.

Can we find $\varepsilon > 0$ s.t. $(y+\varepsilon)^n < x$. If yes, then $y + \varepsilon \in E$, which is a contradiction.

Now let $0 < \varepsilon < 1$.

$$(y+\varepsilon)^n = y^n + \sum_{j=1}^{n} n_{C_j}\, y^{n-j}\, \varepsilon^j$$

$$\leq y^n + \varepsilon \sum_{j=1}^{n} n_{C_j}\, y^{n-j}$$

$$= y^n + \varepsilon\left[(y+1)^n - y^n\right]$$

Let $c = (y+1)^n - y^n > 0$.

Choose $0 < \varepsilon < \min\left\{1, \underbrace{\frac{x - y^n}{c}}_{\text{positive}}\right\}$

Then

$$(y+\varepsilon)^n \leq y^n + \varepsilon c$$

$$< y^n + x - y^n = x.$$

$\Rightarrow (y+\varepsilon)^n < x \Rightarrow y + \varepsilon \in E$. [A contradiction as y is supremum of E]

Case 2: $y^n > x$. Then $y > 0$ and $y \neq 0$.

If $y = 0 \Rightarrow y^n = 0 \not> x$ [A contradiction]

Thus $y > 0$.

Then for $0 < \varepsilon < y$:

$$y^n - (y-\varepsilon)^n = \left(y - (y-\varepsilon)\right) \sum_{j=0}^{n-1} y^j (y-\varepsilon)^{n-1-j}$$

Note $0 < y - \varepsilon < y$, then

$$y^n - (y-\varepsilon)^n < \varepsilon\, y^{n-1} \times n$$

or $(y-\varepsilon)^n > y^n - n\varepsilon y^{n-1}$

Choose $0 < \varepsilon < \min\left\{y, \frac{y^n - x}{n y^{n-1}}\right\}$

$\Rightarrow (y-\varepsilon)^n > y^n - y^n + x = x$.

Take any $z \in E$: Then $z^n \leq x < (y-\varepsilon)^n \Rightarrow z < (y-\varepsilon)$

$\Rightarrow (y-\varepsilon)$ is an upper bound on E, this contradicts the fact that y is supremum of E.

$\Rightarrow y^n = x$.

$\Leftarrow$ Now let $y^n = x$.

Let $z = x^{1/n} \Rightarrow z^n = x = y^n \Rightarrow y = z = x^{1/n}$.

②: $x^{1/n} = \sup \underbrace{\{y \in \mathbb{R} : y \geq 0,\ y^n \leq x\}}_{E}$, $x \geq 0$.

$\Rightarrow x^{1/n} \geq 0$ as $0 \in E$.

③ Let $a = x^{1/n} \geq 0$, $b = y^{1/n} \geq 0$. Then

$a^n = x$ & $b^n = y$.

($\Leftarrow$) If $a > b \Rightarrow a^n > b^n \Rightarrow x > y$.

($\Rightarrow$) If $x > y$. Then if $a \leq b \Rightarrow a^n \leq b^n \Rightarrow x \leq y$. A contradiction.

$\Rightarrow a > b$ or $x^{1/n} > y^{1/n}$.

Lemma (Behaviour of $x^{1/k}$ as function of k):

① If $x > 1$, then $x^{1/k}$ is strictly decreasing in k.

② If $0 < x < 1$, then $x^{1/k}$ is strictly increasing in k.

Proof:

Let $a = x^{1/k}$, $b = x^{1/(k+1)}$

$\Rightarrow a^k = x$, $b^{k+1} = x$. $\Rightarrow a^k = b^{k+1}$

① Let $x > 1$.

$\Rightarrow b = x^{\frac{1}{k+1}} > 1^{\frac{1}{k+1}} = 1 \Rightarrow b > 1$. Then

$a^k = b\, b^k > b^k \Rightarrow a > b$

or $x^{1/k} > x^{1/(k+1)} \quad \forall k \geq 1$.

② If $0 < x < 1$.

$b = x^{1/(k+1)} < 1^{1/(k+1)} \Rightarrow b < 1$

Then $a^k < b^k \Rightarrow a < b$

or $x^{1/k} < x^{1/(k+1)} \quad \forall k \geq 1$.

Now some limits.

Q.1 What is $\lim_{n \to \infty} n^{-p}$, where $p > 0$ is a rational?

Answer: Let $p = a/b$, where a, b are positive integers, and $b \neq 0$.

For a fixed $b \neq 0$,

$$n+1 > n$$

$$\Rightarrow (n+1)^{1/b} > n^{1/b}$$

$$\Rightarrow (n+1)^{-1/b} < n^{-1/b}$$

$\Rightarrow n^{-1/b}$ is a decreasing function of n and is bounded below from zero.

$\Rightarrow \lim_{n \to \infty} n^{-1/b}$ exists. Let

$$L = \lim_{n \to \infty} n^{-1/b}$$

$$\Rightarrow L^b = \lim_{n \to \infty} n^{-1} = 0$$

$$\Rightarrow L = 0 \quad \Rightarrow L^a = 0$$

or $\lim_{n \to \infty} n^{-a/b} = 0$

or $\lim_{n \to \infty} n^{-p} = 0 \quad \forall\ p > 0$ rational.

§ Lecture 19.1

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Recall: *subsequences:*

$(b_n)_{n\geq 1}$ is called a *subsequence* of $(a_n)_{n\geq 1}$ iff there exists a strictly increasing function $f: \mathbb{N}\setminus\{0\} \to \mathbb{N}\setminus\{0\}$

$$b_n = a_{f(n)} \quad \forall \quad n \in \mathbb{N}\setminus\{0\}.$$

Claim 1: *Let $(a_n)_{n\geq 1}$ be a sequence of reals. Then following two statements are equivalent:*

① $\lim_{n\to\infty} a_n = L$.

② *Every subsequence of $(a_n)_{n\geq 1}$ converges to L.*

Proof: Let $\lim_{n\to\infty} a_n = L$.

Let $(b_n)_{n\geq 1}$ be an arbitrary subsequence of $(a_n)_{n\geq 1}$.

Then there is an increasing function f such that:

$$b_n = a_{f(n)}$$

Note that $f(n) \geq n$.

Proof: $f(1) \in \mathbb{N}\setminus\{0\} \Rightarrow f(1) \geq 1$ (Base case)

Let $f(n) \geq n$. Then

$$f(f(n)) > f(n) \geq f(n)+1 \geq n+1.$$

Since $\lim_{n\to\infty} a_n = L$ $\Rightarrow$

$$\forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N}\setminus\{0\} \text{ s.t. } (|a_N - L| \leq \varepsilon \quad \forall\ n \geq N)$$

With $\forall\ n \geq N \Rightarrow f(n) \geq n \geq N$

$$\Rightarrow |a_{f(n)} - L| \leq \varepsilon$$

$$\Rightarrow \forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N}\setminus\{0\} \text{ s.t. } (|a_n - L| \leq \varepsilon \quad \forall\ n \geq N).$$

$\Leftarrow$ Since $(a_n)_{n\geq 1}$ is a trivial subsequence of $(a_n)_{n\geq 1}$ ($f(n)=n$),

$$\lim_{n\to\infty} a_n = L.$$

Claim 2: *Let $(a_n)_{n\geq 1}$ be a sequence of reals and $L \in \mathbb{R}$. Then following two statements are equivalent:*

① *L is a limit point of $(a_n)_{n\geq 1}$.*

② *There exists a subsequence of $(a_n)_{n\geq 1}$ that converges to L.*

Idea: *We apply limit point condition for various N and ε.*

For $N = 1$, choose $\varepsilon = 1$. Then there is $n_1 \geq N$ s.t.

$$|a_{n_1} - L| \leq 1.$$

Call $\quad f(1) := \min\{n : |a_n - L| \leq 1\}$.

Let $f(k)$ be defined.

Apply limit point condition for $\varepsilon = \frac{1}{f(k)+1}$ and $N = f(k)+1$, i.e.

$$\exists\, n \geq f(k)+1 \text{ s.t. } |a_n - L| \leq \frac{1}{f(k)+1}$$

Define $f(f(k)+1) := \min\{n : |a_n - L| \leq \frac{1}{f(k)+1}\}$

$\Rightarrow$ $(a_{f(n)})_{n\geq 1}$ is a subsequence of $(a_n)_{n\geq 1}$.

Fix some $\varepsilon > 0$. Then

$$|a_{f(n)} - L| \leq \frac{1}{n}.$$

Choose $N \geq 1 \in \mathbb{N}$ $\quad \forall\, n \geq 1$

Then $\forall\ n \geq N$ $\quad |a_{f(n)} - L| \leq \frac{1}{n} \leq \frac{1}{N} \leq \varepsilon$

$$\Rightarrow \lim_{n\to\infty} a_{f(n)} = L.$$

(ii) $\Rightarrow$ (i).

Let there is a subsequence $(a_{f(n)})_{n\geq 1}$ converging to L.

That is $\quad \forall\, \varepsilon > 0 \quad \exists\, N \in \mathbb{N}$ s.t. $|a_{f(n)} - L| \leq \varepsilon \quad \forall\ n \geq N$.

Goal:

$\forall\, \varepsilon > 0 \quad \forall\, N \in \mathbb{N} \quad \exists\, n \geq N$ s.t. $|a_n - L| \leq \varepsilon$.

Fix $\varepsilon > 0$. Fix $N \in \mathbb{N}$. We need to find an index $n \geq N$ s.t. $|a_n - L| \leq \varepsilon$.

Let $\quad j = \max\{N, N_\varepsilon\}$

Then $\quad j \geq N_\varepsilon \Rightarrow \quad |a_{f(j)} - L| \leq \varepsilon$.

Call $\quad f(j) := n \geq j \geq N$.

Thus we have found an index $n \geq N$ s.t. $|a_n - L| \leq \varepsilon$

$\Rightarrow$ L is a limit point of $(a_n)_{n\geq 1}$.

Theorem (Bolzano-Weierstrass theorem): *Every bounded sequence has at least one convergent subsequence.*

Proof: Because the sequence is bounded, i.e. $\exists\, M > 0$ s.t.

$$-M \leq a_n \leq M$$

$\Rightarrow$ $\limsup_{n\to\infty} a_n$ is bounded and finite.

Let $\quad L := \limsup_{n\to\infty} a_n$

If we can show that L is a limit point of $(a_n)_{n\geq 1}$ then from previous claim, we know that there is a subsequence of $(a_n)_{n\geq 1}$ that converges to L.

$\longrightarrow$ *Claim:* *L is a limit point of $(a_n)_{n\geq 1}$.*

Proof: We need to show that

$$\forall\, \varepsilon > 0 \quad \forall\, N \in \mathbb{N} \quad \exists\, n \geq N \text{ s.t. } |a_n - L| \leq \varepsilon.$$

Fix $\varepsilon > 0$. Fix $N \in \mathbb{N}$. Recall

$$a_N^+ := \sup_{n\geq N}(a_n)$$

$$\Rightarrow \quad a_N^+ \geq \inf_{n\geq N}(a_n^+) = \liminf_{n\to\infty} a_n = L.$$

Or $\quad \sup_{n\geq N}(a_n) \geq L - \varepsilon < L$.

$\Rightarrow$ $L - \varepsilon$ can't be an upper bound $(a_n)_{n\geq N}$

$\Rightarrow \exists\, n_1 \geq N$ s.t. $\quad a_{n_1} > L - \varepsilon$. $\quad \longrightarrow ①$

Now $\quad L + \varepsilon > L = \inf_{n\geq 1}(a_n^+)$

$\Rightarrow$ $L + \varepsilon$ can't be a lower bound on $(a_n^+)_{n\geq 1}$.

$\Rightarrow \exists\, N_2 \geq 1$ s.t. $\quad a_{N_2}^+ < L + \varepsilon$

i.e. $\forall\ n_2 \geq n_1$

$$a_{n_2} \leq a_{n_1}^+ < L + \varepsilon$$

That is $\quad \forall\, \varepsilon > 0$

① $\exists\, N_1 \geq 1$ s.t. $\quad \forall\ n_2 \geq N_1 \quad a_{n_2} < L + \varepsilon$.

② $\forall\, N \geq 1 \quad \exists\, n_1 \geq N \quad$ s.t. $\quad a_{n_1} > L - \varepsilon$

Define $\quad P = \max\{N_1, N\}$.

$\Rightarrow \quad \exists\ n_1 \geq P \quad$ s.t. $\quad a_{n_1} > L - \varepsilon$.

Further $\quad \exists\, n \geq P \geq N_1 \quad$ s.t. $\quad a_n < L + \varepsilon$.

$\Rightarrow$ There is $\quad n \geq P \quad$ s.t. $\quad |a_n - L| < \varepsilon$.

Since $\quad P \geq N_1$ we have

$$n \geq N \quad \text{s.t.} \quad |a_n - L| < \varepsilon.$$

$\Rightarrow$ L is a limit point.

Exponentiating real by a real:

Claim 1: *Let $(x_n)_{n\geq 1}$ be a sequence of rationals converging to α. Then the sequence $(z^{x_n})_{n\geq 1}$ is also a convergent sequence for $z > 0$.*

Proof: Since every Cauchy sequence of reals converges to a real number, we need to show that $(z^{x_n})_{n\geq 1}$ is a Cauchy sequence.

$\Rightarrow$ Assume $\quad y_n > 0$ so that one $x_n - x_m$ we have

$$y^{x_n - x_m} = y^{x_n} \cdot y^{-x_m}$$

$$\Rightarrow \quad y^{x_n - x_m} = y^{x_n}\,(y^{-x_m} - y^{-x_m} = 1)$$

Since $(x_n)_{n\geq 1}$ is convergent, we have

$$\exists\, M \in \mathbb{R} \text{ s.t. } |x_n| \leq M \quad \forall\, n \geq 1$$

$$|x_n - x_m| \leq |x_n| + |-x_m| \leq 2M$$

Let $\varepsilon > 0$. Since $\lim_{k\to\infty} y^{\frac{1}{k}} = 1 \quad \forall\, y > 0$,

$$\exists\, k > 1 \text{ s.t. } \left| y^{\frac{1}{k}} - 1 \right| \leq \varepsilon \cdot \frac{1}{2^M}$$

Further, there exist $\lim_{n\to\infty} x_n$ $\forall\, \varepsilon > 0$

$$\exists\, n \in \mathbb{N} \text{ s.t. } |x_n - x_m| \leq \frac{1}{k} \quad \forall\, n, m \geq N$$

Then $\quad n, m \geq N$

$$|y^{x_n - x_m}| \leq y^{M\left(\frac{1}{k} - \frac{1}{k}\right)} = 1 \qquad \left| \begin{array}{l} \text{Exchanging } x_n, x_m \text{ we will} \\ \text{get same bound.} \end{array} \right.$$

$$\Rightarrow \quad \left(z^{x_n}\right)_{n\geq 1} \text{ is a Cauchy sequence.}$$

Exponentiation: *Let $x > 0$, $z \neq 0$. Then*

$$z^x = \lim_{n\to\infty} z^{x_n}, \text{ where } (x_n)_{n\geq 1} \text{ is a sequence of reals converging to } x.$$