§ Lecture 18.0

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Claim 5 $\lim_{n\to\infty} a_n = c \iff \liminf a_n = c = \limsup a_n$.

Proof: $\Rightarrow$: Previous class.

$\Leftarrow$: Assume $\liminf a_n = c = \limsup a_n$

Fix $\varepsilon > 0$

Our goal is to show that: $\exists N \geq 1$ s.t.

$c - \varepsilon \leq a_n \leq c + \varepsilon \quad \forall\, n \geq N$.

Note that: ①

$c - \varepsilon < c = \sup_{k\geq 1}\left(a_k^-\right)$

$\Rightarrow c-\varepsilon$ can't be an upper bound (as $\sup$ is the smallest upper bound)

$\Rightarrow \exists N_1 \geq 1$ s.t. $a_{N_1}^- > c - \varepsilon$

Then $\forall\, n \geq N_1$, $\quad a_n \geq \inf_{n\geq N_1} a_n = a_{N_1}^- > c - \varepsilon$

$\Rightarrow \forall\, \varepsilon > 0 \ \exists N_1 \geq 1$ s.t. $\forall\, n \geq N_1$

$a_n > c - \varepsilon$. $\longrightarrow$ ③

② $c + \varepsilon > c = \inf_{k\geq 1}\left(a_k^+\right)$

Since $\inf_{k\geq 1}(a_k^+)$ is the greatest lower bound on $\{a_k^+ : n \geq 1\}$, $c+\varepsilon$ can't be a lower bound.

$\Rightarrow \exists\, n_2 \geq 1$ s.t. $a_{n_2}^+ < c + \varepsilon$

Then $\forall\, n \geq n_2$, $\quad a_n \leq \sup_{n\geq n_2} a_n = a_{n_2}^+ < c + \varepsilon$.

Thus $\forall\, \varepsilon > 0$, $\exists\, n_2 \geq 1$ s.t. $\quad a_n < c + \varepsilon \quad \forall\, n \geq n_2$. $\longrightarrow$ ④

Combining ③ & ④ we get

$\forall\, \varepsilon > 0 \quad \exists\, N = \max\{n_1, n_2\} \geq 1$ s.t.

$|a_n - c| < \varepsilon \qquad \forall\, n \geq N$.

$\Rightarrow \lim_{n\to\infty} a_n = c$.

Lemma (Comparison principle): *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of reals with $a_n \leq b_n$ $\forall\, n \geq 1$. Then*

① $\inf_{n\geq 1}(a_n) \leq \inf_{n\geq 1}(b_n)$ ② $\sup_{n\geq 1}(a_n) \leq \sup_{n\geq 1}(b_n)$

③ $\liminf_{n\to\infty}(a_n) \leq \liminf_{n\to\infty}(b_n)$ & $\limsup_{n\to\infty}(a_n) \leq \limsup_{n\to\infty}(b_n)$.

Proof: ① $b_n \geq a_n \geq \inf_{k\geq 1}(a_n) \quad \forall\, n \geq 1$

$\Rightarrow \inf(b_n)$ is a lower bound on (a_n). Then since all lower bounds are smaller than $\inf(a_n)$, we have

$\inf(a_n) \leq \inf(b_n)$

② $a_n \leq b_n \leq \sup(b_n)$

$\Rightarrow \sup(b_n)$ is an upper bound on (a_n), this means

$\sup(a_n) \leq \sup(b_n)$

③ For ④ Then

$a_n \leq b_n \quad \forall\, n \geq 1$

$\Rightarrow \sup_{k\geq n}(a_k) \leq \sup_{k\geq n}(b_k)$

$\Rightarrow a_n^+ \leq b_n^+ \quad \forall\, n \geq 1$

$\Rightarrow \inf_{n\geq 1}(a_n^+) \leq \inf_{n\geq 1}(b_n^+)$

That is $\limsup(a_n) \leq \limsup(b_n)$.

Squeeze test: *Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ be sequences of real numbers such that*

① $a_n \leq b_n \leq c_n \quad \forall\, n \geq 1$. ② $\lim_{n\to\infty} a_n = l = \lim_{n\to\infty} c_n$

Then $\lim_{n\to\infty} b_n = l$.

Proof 1: Applying comparison principle as well as the fact that

$\lim_{n\to\infty} a_n = l \iff \limsup_{n\to\infty} a_n = l = \liminf_{n\to\infty} a_n$

We have $\quad a_n \leq b_n \leq c_n \Rightarrow \limsup a_n \leq \limsup b_n \leq \limsup c_n$

$\Rightarrow l \leq \limsup b_n \leq l$

$\Rightarrow \limsup b_n = l$

Similarly, $\liminf b_n = l$

Thus $\lim_{n\to\infty} b_n = l$.

Proof 2: Since $\lim_{n\to\infty} a_n = l$

$\Rightarrow \forall\, \varepsilon > 0 \quad \exists N \geq 1$ s.t. $\quad |a_n - l| \leq \varepsilon \quad \forall\, n \geq N_1$. $\longrightarrow$ ①

or $\quad l - \varepsilon \leq a_n \leq l + \varepsilon$.

Since $\lim_{n\to\infty} c_n = l$

$\Rightarrow \forall\, \varepsilon > 0 \quad \exists\, N_2 \geq 1$ s.t. $\quad |c_n - l| \leq \varepsilon \quad \forall\, n \geq N_2$. $\longrightarrow$ ②

or $\quad l - \varepsilon \leq c_n \leq l + \varepsilon$.

$\Rightarrow \quad l - \varepsilon \leq a_n \leq b_n \leq c_n \leq l + \varepsilon \quad \forall\, n \geq \max\{N_1, N_2\}$

$\Rightarrow \lim_{n\to\infty} b_n = l$.

Proof 3: Fails

$a_n \leq b_n \leq c_n$

$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} c_n \qquad \left(\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n\right)$

↑ This doesn't go through as we don't know if $\lim_{n\to\infty} b_n$ exists.

Completeness of reals

Theorem: *Every sequence $(a_n)_{n\geq 1}$ of reals is a Cauchy sequence iff it is convergent.*

Proof: We have shown already that every convergent sequence is Cauchy. We just need to show that every Cauchy sequence is convergent.

Let $(a_n)_{n\geq 1}$ be a Cauchy sequence.

Then $(a_n)_{n\geq 1}$ is a bounded sequence. That is there is a real number M s.t. $\quad -M \leq a_n \leq M$

From comparison principle $\Rightarrow \limsup(a_n)$ and $\liminf(a_n)$ are both bounded by M, hence finite.

Let $\limsup_{n\to\infty} a_n = \bar{L}$ and $\liminf_{n\to\infty} a_n = \underline{L}$.

Our goal is to show that $\bar{L} = \underline{L}$.

Let $\varepsilon > 0$ be an arbitrary real number. Since $(a_n)_{n\geq 1}$ is Cauchy

$\exists\, N \geq 1$ s.t. $\quad a_N - \varepsilon \leq a_n \leq a_N + \varepsilon \quad \forall\, n \geq N$.

Then $\quad a_N - \varepsilon \leq a_n \quad \forall\, n \geq N \Rightarrow a_N - \varepsilon \leq \inf_{n\geq N}(a_n)$

$= a_N^-$

$\leq \sup_{k\geq 1} a_k^- = \underline{L}$

i.e. $\quad a_N - \varepsilon \leq \underline{L}$. $\longrightarrow$ ①

Similarly, $\quad a_N + \varepsilon \geq a_n \ \forall\, n \geq N \Rightarrow a_N + \varepsilon \geq \sup_{n\geq N}(a_n)$

$= a_N^+$

$\geq \inf_{k\geq 1} a_k^+ = \bar{L}$

$\Rightarrow \quad \bar{L} \leq a_N + \varepsilon$ $\longrightarrow$ ②

From ① & ② $\quad \bar{L} - \underline{L} \leq 2\varepsilon$

Further $\bar{L} \geq \underline{L}$, we have $\quad 0 \leq \bar{L} - \underline{L} \leq 2\varepsilon$ $\longrightarrow$ ③

$\Rightarrow \forall\, \varepsilon > 0, \ |\bar{L} - \underline{L}| \leq 2\varepsilon \quad \Leftrightarrow \quad \bar{L} = \underline{L}$.

Remark: ① $|x| < \varepsilon \leq \varepsilon \ \forall\, \varepsilon > 0 \Rightarrow x = 0$

Suppose not i.e. $|x| > 0$. Choose $\varepsilon = \frac{|x|}{2}$

Then $\quad |x| \leq \frac{|x|}{2}$ or $\quad 1 \leq \frac{1}{2}$. [contradiction]

② $\forall\, \varepsilon > 0 \ \exists N \geq 1$ s.t. $\quad |a_n - L| \leq \varepsilon \quad \forall\, n \geq N$.

$\not\Rightarrow \quad a_n \geq L \quad \forall\, n \geq N$. ; why?

(Because N depends on ε)

An example of limit:

Claim: $\lim_{n\to\infty} \sqrt[n]{r} = 0$ $\forall$ rational $r>0$.

Proof: Since $r>0$, write $r = \frac{a}{b}$, where $a,b \in \mathbb{N}\setminus\{0\}$.

$$\begin{aligned} & n+1 > n \\ \Rightarrow\ & (n+1)^b > n^b \\ \Rightarrow\ & (n+1)^b a > n^b a \qquad [\,x>y \Leftrightarrow x^{\frac{1}{b}} > y^{\frac{1}{b}},\ b\neq 0\,] \\ \Rightarrow\ & (n+1)^b \frac{1}{n^b} > n^b \frac{1}{n^b} \\ \Rightarrow\ & \left(\frac{n+1}{n}\right)^b > 1 \\ \Rightarrow\ & \left(\sqrt[n]{\tfrac{n+1}{n}}\right)^b < \sqrt[n]{r} \end{aligned}$$

That $\sqrt[n]{r}$ is a decreasing function of n for all $r>0$.

Thus sequence $(\sqrt[n]{r})_{n\geq 1}$ is decreasing and bounded by zero from below. Thus, $\lim_{n\to\infty} \sqrt[n]{r}$ exists.

Let $L = \lim_{n\to\infty} \sqrt[n]{r} = \lim_{n\to\infty} r^{\frac{1}{2n}}$

$$\begin{aligned} \Rightarrow\ L^b &= \lim_{n\to\infty} r^{\frac{b}{2n}} \\ &= \lim_{n\to\infty} r^{\frac{1}{n}} \\ &= \left[\lim_{n\to\infty} r^{\frac{1}{n}}\right]^a \\ &= L^a \\ &= 0. \end{aligned}$$

$$\Rightarrow \boxed{L=0}$$

Subsequences.

Definition: Let $(a_n)_{n\geq 1}$ and $(b_k)_{k\geq 1}$ be sequences of real numbers. We say that $(b_k)_{k\geq 1}$ is a subsequence of $(a_n)_{n\geq 1}$ iff there exists a function $f: \mathbb{N}\to\mathbb{N}$ which is strictly increasing s.t.

$$b_k = a_{f(k)} \qquad \forall\, n \in \mathbb{N}$$

[Note: f is an injective function otherwise it can't be strictly increasing. It may not be bijective though.]

Example: $(b_k)_{k\geq 1}$, where $b_k = a_{2k}$ is a subsequence of $(a_n)_{n\geq 1}$.

Lemma 1: Let $(a_n)_{n\geq 1}$ be a sequence of reals and $L \in \mathbb{R}$. Then following statements are equivalent:

(i) $\lim_{n\to\infty} a_n = L$

(ii) Every subsequence of $(a_n)_{n\geq 1}$ converge to L.

Proof: (ii) $\Rightarrow$ (i) is trivial as $(a_n)_{n\geq 1}$ is a subsequence of itself. (Choose $f(n)=n$).

Since every subsequence of $(a_n)_{n\geq 1}$ converges to $L \Rightarrow \lim_{n\to\infty} a_n = L$.

(i) $\Rightarrow$ (ii) Given $\lim_{n\to\infty} a_n = L$: $\forall\, \varepsilon>0\ \exists\, N\in\mathbb{N}$ s.t. $|a_n - L| \leq \varepsilon\ \forall\, n \geq N$.

Let $f: \mathbb{N}\to\mathbb{N}$ be an arbitrary strictly increasing function.

Then $f(n) \geq n\ \ \forall\, n \in \mathbb{N}$.

[Big indention:
$f(n) \in \mathbb{N} \Rightarrow f(n) \geq 0$
Induction step: $f(n) \geq n$. Then
$f(n+1) > f(n)$
$\Rightarrow f(n+1) \geq f(n)+1 \geq n+1$]

Thus

$$f(n) \geq N \geq n, \text{ i.e.}$$
$$|a_{f(n)} - L| \leq \varepsilon$$

$\Rightarrow \lim_{n\to\infty} a_{f(n)} = L$ for any strictly increasing function f.

Lemma 2: Let $(a_n)_{n\geq 1}$ be a sequence of reals and $L \in \mathbb{R}$. Then following two statements are equivalent

(i) L is a limit point of $(a_n)_{n\geq 1}$.

(ii) There exists a subsequence of $(a_n)_{n\geq 1}$ that converges to L.

Proof: (ii) $\Rightarrow$ (i) Easy direction.

Suppose $(a_{f(n)})_{n\geq 1}$ be a subsequence of $(a_n)_{n\geq 1}$ such that

$$\lim_{n\to\infty} a_{f(n)} = L$$
$$\Rightarrow \forall \varepsilon>0\ \ \exists\, N_0 \geq 1 \text{ s.t. } |a_{f(n)} - L| \leq \varepsilon \quad \forall\ n \geq N_0 \longrightarrow (*)$$

Goal: $\forall\, \varepsilon>0\ \ \forall N \geq 1\ \ \exists\, m \geq N$ s.t. $|a_m - L| \leq \varepsilon$.

Fix $\varepsilon>0$. Fix $N \geq 1$. We need to find $m \geq N$.

① $m \geq 1$, ② $|a_m - L| \leq \varepsilon$

Pick $j = \max\{f_1, N_0\}$

From (*) $|a_{f(j)} - L| \leq \varepsilon$

Call $f(j) = m$ $\Rightarrow$ ① $m = f(j) \geq j \geq N$

② $|a_m - L| \leq \varepsilon$

$\Rightarrow L$ is a limit point of $(a_n)_{n\geq 1}$.

(i) $\Rightarrow$ (ii) Difficult direction.

Given L is a limit point of $(a_n)_{n\geq 1}$ that is

$\forall\, \varepsilon>0\ \ \forall N \geq 1\ \ \exists\, m \geq N$ s.t. $|a_m - L| \leq \varepsilon$. $\longrightarrow (*)$

Goal: To construct a subsequence that converge to L.

Recursive construction: $N \geq 1$, $\varepsilon \leq 1$

$\exists\, n \geq 1$ s.t. $|a_n - L| \leq 1$.

Call $f(1)$, the smallest n satisfying above.

Recursive step: Let $f(j)$ has been defined.

Apply $(*)$ with $\varepsilon = \frac{1}{j+1}$, $N = f(j)+1$

Then $\exists\, n \geq f(j)+1$ s.t. $|a_n - L| \leq \frac{1}{j+1}$

Let $f(j+1)$ be the smallest such n. [Well defined from well ordering principle]

Thus $f(j)$ is defined for all naturals.

By construction: ① $f(j+1) \geq f(j)+1 > f(j)$

② $|a_{f(j)} - L| \leq \frac{1}{j}$

From Archimedean property choose $\exists\, j \geq 1$

Then $\forall\ j \geq J$ $|a_{f(j)} - L| \leq \frac{1}{J} \leq \varepsilon$.

$\Rightarrow \lim_{j\to\infty} a_{f(j)} = L$.

Recall: ① convergence $\Leftrightarrow$ Cauchy.

② Bounded + monotone $\Rightarrow$ convergence

③ Bounded $\not\Rightarrow$ convergence

Maybe weaken it has limit points $\Rightarrow$ only a subsequence converges !!

Theorem (Bolzano–Weierstrass theorem): Every bounded sequence of reals has at least one convergent subsequence. [Important]

Proof: Given: A real sequence $(a_n)_{n\geq 1}$ such that $\exists\, M>0$ and

$$m \leq a_n \leq M. \qquad \longrightarrow (*)$$

This means that $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ are real (bounded).

Let $L = \liminf_{n\to\infty} a_n$.

If we can show that L is a limit point, then there is a subsequence that converge to L. [Lemma 2 above]

Limit point: $\forall \varepsilon>0\ \ \forall N \geq 1\ \ \exists\, n \geq N$ s.t. $|a_n - L| \leq \varepsilon$.

Fix $\varepsilon>0$. Fix $N \geq 1$.

We have: $L \pm \varepsilon > L = \inf_{M \geq 1} \left(a_M^+\right)$

$\Rightarrow L \pm \varepsilon$ cannot be a lower bound of $(a_M^+)_{M\geq 1}$

$\Rightarrow \exists\ M_0 \geq 1$ s.t. $a_{M_0}^+ < L + \varepsilon$.

That is $\forall\ n \geq M_0$ $a_n \leq a_{M_0}^+ < L+\varepsilon$. $\longrightarrow$ ①

Mind that we is some number independent of N.

Further $L-\varepsilon < L = \inf_{M\geq 1} \left(a_M^+\right)$

$\forall\ M \geq 1$ $a_M^+ > L - \varepsilon$

$\sup_{n \geq M} a_n > L - \varepsilon$

$\Rightarrow L - \varepsilon$ can't be an upper bound. That is $\exists\ m_\varepsilon \geq M$ s.t.

$$a_{m_\varepsilon} > L - \varepsilon.$$

Recalling: $\exists\, N_0 \geq 1$ $\forall\ m \geq N_0$ $a_m < L + \varepsilon$

$\forall\ N \geq 1$ $\exists\ m_\varepsilon \geq N$ $a_{m_\varepsilon} > L - \varepsilon$ $\longrightarrow$ ②

Let $P = \max\{f_1, N\}$

$\exists\ n \geq P$ $a_n > L - \varepsilon$ $\longrightarrow$ ③

$\Rightarrow$ $n \geq P \geq N_0$ $a_n < L + \varepsilon$ $\longrightarrow$ ④

Since $n \geq P \geq N$, we have

$$|a_n - L| < \varepsilon.$$

Thus $\forall \varepsilon>0$, $\forall N \geq 1$ $\exists\, n \geq N$ s.t. $|a_n - L| < \varepsilon$.

Thus L is a limit point of $(a_n)_{n\geq 1}$.