§ Lecture 17.0

Monday, 12 October 2026 17:30

Claim (Limits are limit points) :- Let $\lim_{n\to\infty} a_n = L$. Then L is the only limit point of sequence $(a_n)_{n\geq 1}$.

Proof: Previous class !

Recall: $a_n = (-1)^n [1 + 10^{-n}]$

± 1 are limit points of $(a_n)_{n\geq 1}$. Can we associate a meaning to these two limit points?

Definition (Limit superior and limit inferior) :-

Let $(a_n)_{n\geq 1}$ be a sequence of reals. Define

$$a_N^+ = \sup_{n\geq N} a_n \qquad\qquad a_N^- = \inf_{n\geq N} a_n$$

$$a_1^+ = \sup\{a_1, a_2, a_3, \cdots\} \qquad a_1^- = \inf\{a_1, a_2, a_3, \cdots\}$$

$$a_2^+ = \sup\{a_2, a_3, a_4, \cdots\} \qquad a_2^- = \inf\{a_2, a_3, a_4, \cdots\}$$

$$a_3^+ = \sup\{a_3, a_4, a_5, \cdots\} \qquad a_3^- = \inf\{a_3, a_4, a_5, \cdots\}$$

$$\vdots \qquad\qquad \vdots$$

Then

$$\limsup_{n\to\infty} a_n := \inf_{N\geq 1} a_N^+,$$

$$\liminf_{n\to\infty} a_n := \sup_{N\geq 1} a_N^- .$$

Example: $a_n = (-1)^n [1 + 10^{-n}]$.

Note the following: let $k \in \mathbb{N}\cup\{0\}$

$$a_{2k} = 1 + 10^{-2k} \quad ; \quad a_{2k+2} = 1 + 10^{-2k} \times 10^{-2} \leq a_{2k}$$

$$a_{2k-1} = -[1 + 10\cdot 10^{-2k}] \; ; \; a_{2k+1} = -[1 + 10^{-1}\, 10^{-2k}] \geq a_{2k-1}.$$

That is ① All even terms are positive and decreasing.

② All odd terms are negative and increasing.

Now fix some $N\geq 1$.

$$a_N^+ = \sup_{n\geq N} a_n$$

① If $N = 2k$ for some natural k, then

$$a_N \leq [1 + 10^{-2k}] \quad \forall\, N \geq 2k$$

(Because all even terms are decreasing and odd terms are negative.)

Since $1 + 10^{-2k} \in \{a_N : N \geq 2k\}$

$$\Rightarrow \quad a_N^+ = 1 + 10^{-2k} \quad \text{, when } N = 2k.$$

② If $N = 2k-1$, then

$$a_N \leq 1 + 10^{-2k} \quad [\text{same logic}]$$

$$\Rightarrow \quad a_N^+ = 1 + 10^{-2k}, \quad \text{when } N = 2k-1.$$

Thus,

$$\left.\begin{aligned} a_1^+ &= 1 + 10^{-2} \\ a_2^+ &= 1 + 10^{-2} \\ a_3^+ &= 1 + 10^{-4} \\ a_4^+ &= 1 + 10^{-4} \\ &\vdots \end{aligned}\right\} \Rightarrow \limsup_{n\to\infty} a_n = \inf_{N\geq 1} a_N^+ = \inf\{1+10^{-2}, 1+10^{-4}, \cdots\}$$

Here it is a monotonic decreasing sequence that is lower bounded by 1. Then

$$\inf_{N\geq 1} a_N^+ = \lim_{N\to\infty} a_N^+ = \lim_{N\to\infty} [1 + 10^{-2N}] = 1.$$

Thus $\limsup_{n\to\infty} a_n = 1$.

Now about liminf: Fix some $N\geq 1$.

$$a_N^- = \inf_{n\geq N} a_n.$$

If $N = 2k+1$ for some $k \in \mathbb{N}$, then

$$a_N \geq -[1 + 10^{-(2k+1)}] \qquad (N \geq 2k+1)$$

$$\Rightarrow \quad a_N^- = -[1 + 10^{-(2k+1)}] \quad \text{if } N = 2k+1.$$

If $N = 2k$, then

$$a_N \geq -[1 + 10^{-(2k+1)}] \qquad (N \geq 2k)$$

$$\Rightarrow \quad a_N^- = -[1 + 10^{-(2k+1)}] \quad \text{if } N = 2k$$

$$\Rightarrow \quad a_1^- = -[1 + 10^{-1}]$$

$$a_2^- = -[1 + 10^{-3}]$$

$$a_3^- = -[1 + 10^{-3}]$$

$$a_4^- = -[1 + 10^{-5}]$$

$$\vdots$$

$$\sup_{N\geq 1} a_N^- = \sup\{-[1 + 10^{-(2k+1)}] : k \geq 0\}$$

$-[1 + 10^{-(2k+1)}]$ is an increasing sequence upper bounded by 1.

$$\Rightarrow \quad \sup_{N\geq 1} a_N^- = \lim_{k\to\infty} -[1 + 10^{-(2k+1)}]$$

$$= -1$$

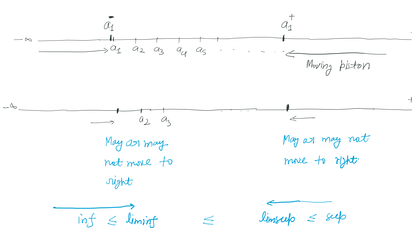
$$\Rightarrow \quad \liminf_{n\to\infty} a_n = -1$$

Summary: $a_n = (-1)^n [1 + 10^{-n}]$

① Not convergent.

② ± 1 and no other numbers are limit points.

③ $\limsup a_n = 1$, $\liminf a_n = -1$.



$\inf \leq \liminf \leq \limsup \leq \sup$

Claim 1: Let $(a_n)_{n\geq 1}$ be a sequence of reals.

(a) For every $x > \limsup a_n$, $\exists\, N\geq 1$ s.t. $a_n < x \;\forall\, n\geq N$.

(b) For every $y < \liminf a_n$, $\exists\, N\geq 1$ s.t. $a_n > y \;\forall\, n\geq N$.

Proof: Suppose $x > \limsup(a_n) = \inf_{N\geq 1}(a_N^+)$

Now $\exists\, N\geq 1$ s.t. $x > a_N^+$ otherwise $\forall N\geq 1$, $x \leq a_N^+$, that is

x is a lesser bound larger than $\inf a_n^+$

[contradiction]

That is there exist $N\geq 1$ s.t. $x > a_N^+ = \sup\{a_n : n\geq N\}$

$\Rightarrow x > a_n \;\forall\; n\geq N$. (Eventually greater than all a_n)

(b) $y < \liminf(a_n) = \sup_{N\geq 1}(a_N^-)$

$\Rightarrow \exists N\geq 1$ s.t. $y < a_N^-$ otherwise $\forall N\geq 1$, we have $y \geq a_N^-$.

(That is we have an upbound smaller than supremum, a contradiction)

$\Rightarrow \exists N\geq 1$ s.t. $y < a_N^- = \inf\{a_n : n\geq N\}$

$\Rightarrow y < a_n \;\forall\; n\geq N$.

Claim 2: (a) For every $x < \limsup(a_n)$ $\forall\; N\geq 1$ $\exists\, n\geq N$ s.t. $x < a_n$.

(b) For every $y > \liminf(a_n)$ $\forall N\geq 1$ $\exists\, n\geq N$ s.t. $y > a_n$.

Proof: $x < \limsup(a_n) = \inf_{N\geq 1}(a_N^+)$

Fix any $N\geq 1$. Then $a_N^+ \geq \inf\{a_n^+\} > x$

i.e. $x < \sup\{a_n : n\geq N\}$

$\Rightarrow \exists\, n\geq N$ s.t. $x < a_n$ [Otherwise $\forall n\geq N$ $a_n \leq x \Rightarrow x$ is an upbound smaller than sup]

Similarly, $y > \liminf a_n = \sup_{N\geq 1}(a_N^-)$

$\Rightarrow \forall N\geq 1$ $a_N^- \leq \sup_{N\geq 1}(a_N^-) < y$

$\Rightarrow y > \inf_{n\geq N}(a_n) \Rightarrow \exists\, n\geq N$ s.t. $y > a_n$

Claim 3: $\inf_{n\geq 1}(a_n) \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup_{n\geq 1}(a_n)$

Proof: $\sup_{n\geq 1} a_n = a_1^+$

And $\limsup_{n\to\infty} a_n = \inf\{a_n^+ : n\geq 1\}$

$\leq a_1^+ = \sup_{n\geq 1}(a_n)$

Similarly, $\liminf_{n\to\infty} a_n = \sup_{N\geq 1}(a_N^-)$

$\geq a_1^- = \inf_{n\geq 1}(a_n)$

The middle part:

Let $N, M \geq 1$. Take $K = \max\{N, M\}$

$\Rightarrow K\geq N$ & $K\geq M$.

Then $a_K \leq \sup_{n\geq M}(a_n) = a_M^+$

$a_K \geq \inf_{n\geq N}(a_n) = a_N^-$

$\Rightarrow a_N^- \leq a_K \leq a_M^+$

or $a_N^- \leq a_M^+$

Fix M, and vary $N\geq 1$. $\Rightarrow a_M^+$ is an upper bound on $\{a_N^- : N\geq 1\}$.

$\Rightarrow \sup_{N\geq 1} a_N^- \leq a_M^+$

Similarly, $\sup_{N\geq 1} a_N^-$ is a lesser bound on $\{a_M^+ : M\geq 1\}$.

$\Rightarrow \sup_{N\geq 1} a_N^- \leq \inf_{M\geq 1} a_M^+$

$\Rightarrow \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$.

Thus $\inf_{n\geq 1} a_n \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \sup_{n\geq 1} a_n$

Claim 4: If c is any limit point of $(a_n)_{n\geq 1}$ then $\liminf a_n \leq c \leq \limsup a_n$.

Proof: Given c is a limit point. That is

$\forall \varepsilon > 0$ $\forall\, N\geq 1$ $\exists\; n\geq N$ $|a_n - c| \leq \varepsilon$. $\longrightarrow (1)$

$\Rightarrow a_n \geq c - \varepsilon$

That is $a_N^+ \geq c - \varepsilon$ [Otherwise a_N^+ cannot be an upbound]

Since this is true for all ε, we have

$a_N^+ \geq c$ [Otherwise if $a_N^+ < c$, then choose $\varepsilon = \frac{c - a_N^+}{2} > 0$ to get $a_N^+ \geq \frac{c + a_N^+}{2} > a_N^+$]

Since $a_N^+ \geq c$ $\forall N\geq 1$ $\Rightarrow c$ is a lesser bound. Then $c \leq \inf\{a_N^+\}$

$\Rightarrow c \leq \limsup a_n$.

Now other part: we have $\forall \varepsilon > 0$ $\forall N\geq 1$ $\exists\, n\geq N$ s.t. $a_n \leq c + \varepsilon$.

$\Rightarrow a_N^- \leq c + \varepsilon$ (Otherwise a_N^- can't be lesser bound as there is an smaller element than $c+\varepsilon$)

$\forall \varepsilon > 0$.

$\Rightarrow a_N^- \leq c$.

$\Rightarrow \sup_{N\geq 1} a_N^- \leq c$

or $\liminf_{n\to\infty} a_n \leq c$.

Combining $\liminf_{n\to\infty} a_n \leq c \leq \limsup_{n\to\infty} a_n$.

Claim 5: $(a_n)_{n\geq 1}$ converges to c iff $\limsup_{n\to\infty} a_n = c = \liminf_{n\to\infty} a_n$.

Proof: Let $\limsup a_n = c = \liminf a_n$

Let $\varepsilon > 0$. Then

① $c + \varepsilon > c = \limsup_{n\to\infty} a_n$

$\Rightarrow \exists N_2 \geq 1$ s.t. $\forall\, n \geq N_2$ $c + \varepsilon > a_n$.

② $c - \varepsilon < c = \liminf_{n\to\infty} a_n$

$\Rightarrow \exists N_1 \geq 1$ s.t. $\forall\, n \geq N_1$ $c - \varepsilon < a_n$.

Take $N = \max\{N_1, N_2\}$ then

$c - \varepsilon < a_n < c + \varepsilon$ $\forall n \geq N$

$\Rightarrow \boxed{\lim_{n\to\infty} a_n = c}$

$\Leftarrow$ (other direction) Let $\lim_{n\to\infty} a_n = c$.

$\forall \varepsilon > 0$ $\exists N \geq 1$ s.t. $\forall n \geq N$ $c - \varepsilon \leq a_n \leq c + \varepsilon$

Now $a_n \leq c + \varepsilon \Rightarrow a_N^+ \leq c + \varepsilon$

$\limsup_{n\to\infty} a_n = \inf_{k\geq 1}(a_k^+) \leq a_N^+ \leq c + \varepsilon$

Similarly $a_n \geq c - \varepsilon$ $n \geq N$

Fix any $k \geq 1$ there is a term $a_{\max(k,N)}$ in a_k^+

Then $a_k^+ \geq c - \varepsilon$ $\forall\, k$

$\Rightarrow \inf a_k^+ \geq c - \varepsilon$

Thus $\left|\limsup_{n\to\infty} a_n - c\right| \leq \varepsilon$ $\forall \varepsilon > 0$.

$\Rightarrow \boxed{\limsup_{n\to\infty} a_n = c}$

Similarly $\liminf_{n\to\infty} a_n = c$.