§ Lecture 16.0

Thursday, 8 October 2026 19:40

Handling a bit of infinity.

We already have defined that for set E with no upper bound

$$\sup(E) = +\infty$$

Similarly $\sup \phi = -\infty$.

Issues:

① We can't compare them yet!

② If we want to define $\sup\{\sup_i E_i\}$, then we run into trouble as sup is only defined for set of real numbers and $\pm\infty$ are not real number.

Extended real number system.

$$\mathbb{R}^* = \mathbb{R} \cup \{+\infty, -\infty\}$$

A real number x is called finite if $x \in \mathbb{R}$ otherwise infinite.

Negation of extended reals.

Let $x \in \mathbb{R}^*$ then if $x \in \mathbb{R}$, $-x$ is usual negation. Further $-(+\infty) = -\infty$, $-(-\infty) = +\infty$.

We donot define any other algebraic operation on $\pm\infty$.

Ordering of extended reals.

Let $x, y \in \mathbb{R}^*$. We say that $x \le y$ iff one of the following three statements is true.

(i) $x, y \in \mathbb{R}$ and $y - x \ge 0$. [Usual]

(ii) $x = -\infty$

(iii) $y = +\infty$

That is $-\infty < y$ for any $y \in \mathbb{R}^*$. In particular $-\infty \le +\infty$.
(Distinct, so $-\infty < +\infty$.)

Supremum of the subsets of the extended reals.

Let $E \subseteq \mathbb{R}^*$. $\sup(E)$ is defined as follows.

(a) If $E \subseteq \mathbb{R}$, then $\sup(E)$ is usual definition. [$+\infty, -\infty$ can appear as answer]

(b) If $+\infty \in E$ then $\sup(E) = +\infty$.

(c) If $+\infty \notin E$ and $-\infty \in E$, then $\sup(E) = \sup(E - \{-\infty\})$ [on usual $\mathbb{R}$]

Similarly $\inf(E) = -\sup(-E)$.

Supremum and infima of sequences.

Let $(a_n)_{n\ge1}$ be a sequence of real numbers. Then

$$\sup(a_n)_{n\ge1} \equiv \sup_{n\ge1} a_n := \sup\{a_n : n \ge 1\}$$

$$\inf(a_n)_{n\ge1} \equiv \inf_{n\ge1} a_n := \inf\{a_n : n \ge 1\}.$$

Monotone sequences.

Let $(a_n)_{n\ge1}$ be a sequence. It is said to be

(i) Monotone increasing if $a_{n+1} \ge a_n \quad \forall\ n \ge 1$.

(ii) Monotone decreasing if $a_{n+1} \le a_n \quad \forall\ n \ge 1$

We say that a sequence is monotone if it is either monotone incresing or monotone decreasing.

Lemma. (Monotone bounded sequences converge).

Let $(a_n)_{n\ge1}$ be a sequence of real numbers.

(i) If $(a_n)_{n\ge1}$ is monotone increasing and upper bounded by $M_0 \in \mathbb{R}$, then

$$\lim_{n\to\infty} a_n = \sup_{n\ge1} a_n \le M_0 \quad [(a_n)_{n\ge1} \text{ is convergent}]$$

(ii) If $(a_n)_{n\ge1}$ is monotone decreasing and lower bounded by $M_1 \in \mathbb{R}$, then

$$\lim_{n\to\infty} a_n = \inf_{n\ge1} a_n \ge M_1.$$

Proof. ① Given $a_{n+1} \ge a_n \quad \forall\ n \ge 1$

$a_n \le M \quad \forall\ n \ge 1$

Since we don't know the limit exists we can't take limit both sides.

Let $s = \sup_n a_n$. Our goal is to show that $(a_n)_{n\ge1}$ converges to s.

Fix $\varepsilon > 0$.

Note that $s \le M$.

Since any upper bound on $(a_n)_{n\ge1}$ must be larger than s, $s-\varepsilon$ can't be an upper bound.

$\Rightarrow \exists\ N \ge 1$ s.t. $a_N > s - \varepsilon$.

But $a_{n+1} \ge a_n > s - \varepsilon$. That is $\forall\ n \ge N \quad a_n > s - \varepsilon$.

Now $\forall\ n \ge N$

$$s - \varepsilon < a_n \le s < s + \varepsilon$$

$$\Rightarrow |a_n - s| < \varepsilon$$

That is $(a_n)_{n\ge1}$ is convergent and $\lim_{n\to\infty} a_n = \sup_n(a_n) \le M$.

(ii) Let $\inf_n a_n = L \ge M_1$.

Fix $\varepsilon > 0$. Then since every lower bound is smaller than L, $L+\varepsilon$ can't be a lower bound. That is $\exists\ N \ge 1$ s.t.

$$a_N < L + \varepsilon \qquad (\text{Then } a_{N+1} \le a_N < L + \varepsilon)$$

Then $\forall\ n \ge N$

$$L - \varepsilon < L \le a_n < L + \varepsilon \Rightarrow |a_n - L| < \varepsilon$$

$$\Rightarrow \lim_{n\to\infty} a_n = \inf_{n\ge1} a_n.$$

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Example: Let $0<x<1$. Then $\lim_{n\to\infty} x^n = 0$.

Solution: Take the sequence $(x^n)_{n\geq 1}$.

This is a decreasing sequence: $x^{n+1} = x\,x^n < x^n$.

Since $x>0 \Rightarrow x^n > 0 \quad \forall\, n\geq 1$.

Then $\lim_{n\to\infty} x^n = L$ (From convergence of monotone bounded sequences).

Further $\lim_{n\to\infty} x^n = \lim_{n\to\infty} x^{n+1}$ $\quad [(x, x^2, \cdots)^{(1)}, (x^2, x^3, \cdots)^{(2)}]$

$= \lim_{n\to\infty} x\,x^n$

$L = xL \Rightarrow L(x-1) = 0$.

But $x\neq 1 \Rightarrow L = 0$.

Thus for $0<x<1$, $\lim_{n\to\infty} x^n = 0$.

Story: We are talking about limits of sequences or convergence.

That is a sequence could be either convergent or divergent.

Is there something more to divergent sequences?

e.g. $a_n = (-1)^n(1+10^{-n}) \quad n\geq 1$

Claim: This sequence is not convergent.

Proof: Note that every convergent sequence is cauchy.

Contrapositive: If a sequence is not Cauchy, it is not convergent.

Then $|a_n - a_{n+1}| = |(-1)^n[1+10^{-n} + 1 + 10^{-n}]|$

$= |2 + 2\times 10^{-n}|$

> 2 (For every n).

Choose $\varepsilon = 1$. then $\forall\, n\geq 1 \quad \exists\, j\geq n,\ k = n+1$ s.t.

$|a_j - a_k| > 2 > \varepsilon$

$\Rightarrow (a_n)_{n\geq 1}$ is not Cauchy and it is not convergent.

In particular $+1$ or -1 cannot be limits of $(a_n)_{n\geq 1}$.

$-1.1,\ 1.01,\ -1.001,\ 1.0001,\ \cdots$ (This sequence doesn't seem to settle at one limit. However, it wants to settle at both ± 1.)

Can we associate some meaning to ± 1?

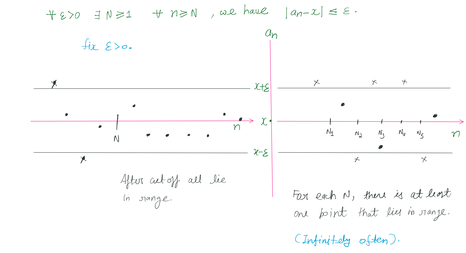
Definition (Limit points):

Let $(a_n)_{n\geq 1}$ be a sequence of real numbers. Let $x\in\mathbb{R}$. Then x is called a limit point of $(a_n)_{n\geq 1}$ iff

$\forall\, \varepsilon > 0 \quad \forall\, N\geq 1 \quad \exists\, n\geq N$ s.t. $|a_n - x| \leq \varepsilon$.

Compare to convergence:

$\forall\, \varepsilon > 0 \quad \exists\, N\geq 1 \quad \forall\, n\geq N$, we have $|a_n - x| \leq \varepsilon$.



Ex: $a_n = (-1)^n[1+10^{-n}] \quad n\geq 1$. ± 1 both are limit points of $(a_n)_{n\geq 1}$.

Fix $\varepsilon > 0$. Fix $N\geq 1$ as an arbitrary integer. Our goal is to find $n\geq N$ s.t. $|a_n - 1| \leq \varepsilon$.

First note that $10^n \geq n \quad \forall\, n\geq 1$ (Induction).

Given N and ε, choose integer $k\geq N$ and $k\in\mathbb{N}, k>1$.

Take $n = 2k > k \geq N$, Then $|a_{2k} - 1| = 10^{-2k} \leq \frac{1}{2k} < \varepsilon$

Similarly for -1, choose $n = 2k+1 > k \geq N$

$|a_{2k+1} + 1| = 10^{-(2k+1)} \leq \frac{1}{2k+1} \leq \frac{1}{2k} < \varepsilon$

Thus ± 1 both are limit points.

In fact no other point x, i.e. $x\neq \pm 1$ is a limit point.

Define $d = \min\{|x-1|, |x+1|\} > 0$.

$\Rightarrow |(-1)^n - x| \geq d \quad \forall\, n$.

Choose $\varepsilon = d/3$, choose N s.t. $N\in\mathbb{Z}, N\geq 3$. (These are chosen, i.e. $\exists\,\varepsilon, \exists N$)

$|a_n - (-1)^n| = 10^{-n} \leq \frac{1}{n} \leq \frac{1}{N} \leq \frac{d}{3} = \varepsilon \quad \forall\, n\geq N$.

Further $d \leq |(-1)^n - x| \leq |(-1)^n - a_n| + |a_n - x|$

$\leq d/3 + |a_n - x|$

$\Rightarrow |a_n - x| \geq 2d/3 > d/3 = \varepsilon$.

For this $\varepsilon = d/3$, and this N, $\forall\, n\geq N \quad |a_n - x| > \varepsilon$.

$\Rightarrow x$ is not a limit point. $[\forall\varepsilon\ \forall N\ \exists n\geq N \ |a_n - x| \leq \varepsilon]$: Recall: x being a limit point

Claim (Limits are limit points): Let $(a_n)_{n\geq 1}$ be a sequence converging to $L\in\mathbb{R}$. Then L is the only limit point of $(a_n)_{n\geq 1}$.

Proof: $\forall\, \varepsilon > 0 \quad \exists\, N_0 \geq 1$ s.t. $\forall\, n\geq N \quad |a_n - L| \leq \varepsilon$.

Fix an $\varepsilon > 0$. Let $\tilde{N} \geq 1$ be arbitrary.

Define $N_1 = \max\{N_0, \tilde{N}\}$

$\Rightarrow N_1 \geq \tilde{N}$, and $N_1 \geq N_0 \Rightarrow |a_{N_1} - L| \leq \varepsilon$.

Thus for each $\tilde{N}$, there is N_1 s.t. $|a_{N_1} - L| \leq \varepsilon$.

$\Rightarrow L$ is a limit point.

Suppose L' is also a limit point. with $L'\neq L$.

Let $\varepsilon_0 = \frac{|L-L'|}{3}$

Given $\exists N_0 \geq 1$ s.t. $\forall\, n\geq N_0 \quad |a_n - L| \leq \varepsilon_0$. (Convergence)

Further, $\exists\, n_1 \geq N_0$ s.t. $|a_n - L'| \leq \varepsilon_0$ (Limit point)

$\Rightarrow |L-L'| \leq |L - a_{n_1}| + |L' - a_{n_1}|$

$\leq \frac{2|L-L'|}{3}$ (Contradiction)

Thus $L = L'$.