§ Lecture 11.0

Wednesday, 16 September 2026 18:19

Meaning of well defined operations: Suppose we have a collection of objects of the form $\underset{n\to\infty}{\mathrm{LIM}}(a_n)$, where $(a_n)_{n\geq 1}$ is a Cauchy sequence of rationals. Let us call this collection as $\mathbb{R}$.

(Collection of real numbers)

And $\underset{n\to\infty}{\mathrm{LIM}}(a_n) = \underset{n\to\infty}{\mathrm{LIM}}(b_n)$, whenever Cauchy sequence $(a_n)_{n\geq 1}$ is equivalent to Cauchy sequence $(b_n)_{n\geq 1}$

That is different names "$\underset{n\to\infty}{\mathrm{LIM}}\, a_n$" and "$\underset{n\to\infty}{\mathrm{LIM}}\, b_n$" are the one and same real number as long as $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are equivalent.

On such a collection the operations $op_1 : \mathbb{R}\to\mathbb{R}$ and $op_2 : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are well defined if

(1) (a) $\forall\, x\in\mathbb{R}$ $(x = \underset{n\to\infty}{\mathrm{LIM}}\, a_n)$ $\quad op_1(x)\in\mathbb{R}$.

(b) $\forall\, x, y\in\mathbb{R}$ $\quad op_2(x,y)\in\mathbb{R}$

(2) (a) $\forall\, x = y\in\mathbb{R}$ $\quad op_1(x) = op_1(y)$

(b) $\forall\, x = y\in\mathbb{R}$ and $z\in\mathbb{R}$ $\quad op_2(x,z) = op_2(y,z)$.

$\Big\}$ substitution property.

Addition and multiplication: These are operations on $\mathbb{R}\times\mathbb{R}$. Let

$x = \underset{n\to\infty}{\mathrm{LIM}}(a_n)$, $y = \underset{n\to\infty}{\mathrm{LIM}}\, b_n$, where $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences on rationals.

Then

(1) $x + y = \underset{n\to\infty}{\mathrm{LIM}}\, a_n + \underset{n\to\infty}{\mathrm{LIM}}\, b_n := \underset{n\to\infty}{\mathrm{LIM}}(a_n + b_n)$

(2) $x\cdot y = \left(\underset{n\to\infty}{\mathrm{LIM}}\, a_n\right)\left(\underset{n\to\infty}{\mathrm{LIM}}\, b_n\right) := \underset{n\to\infty}{\mathrm{LIM}}(a_n b_n)$.

For these operations to be well defined, we need to show that

(1) If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences then

(i) $(a_n + b_n)_{n\geq 1}$ is also a Cauchy sequence.

(ii) $(a_n b_n)_{n\geq 1}$ is also a Cauchy sequence.

(2) [Substitution] If $\underset{n\to\infty}{\mathrm{LIM}}\, a_n = \underset{n\to\infty}{\mathrm{LIM}}\, b_n$, then

(i) $\underset{n\to\infty}{\mathrm{LIM}}(a_n + c_n) = \underset{n\to\infty}{\mathrm{LIM}}(b_n + c_n)$, where $(c_n)_{n\geq 1}$ is a Cauchy sequence.

That is, if $(a_n)\sim(b_n)$, then $(a_n + c_n)\sim(b_n + c_n)$

(ii) $(a_n c_n)\sim(b_n c_n)$.

Proof: Given $(a_n), (b_n), (c_n)$ are Cauchy sequences. Then

$\forall\, \varepsilon_1 > 0\ \exists\, N_1\geq 1$ s.t. $|a_m - a_n|\leq\varepsilon_1\ \forall\, m,n\geq N_1 \longrightarrow (1)$

$\forall\, \varepsilon_2 > 0\ \exists\, N_2\geq 1$ s.t. $|b_m - b_n|\leq\varepsilon_2\ \forall\, m,n\geq N_2 \longrightarrow (2)$

$\forall\, \varepsilon_3 > 0\ \exists\, N_3\geq 1$ s.t. $|c_m - c_n|\leq\varepsilon_3\ \forall\, m,n\geq N_3 \longrightarrow (3)$

Since Cauchy sequences are bounded, we have

$\exists\, M_1\geq 0$ s.t. $|a_n|\leq M_1 < M_1 + 1\ \forall\, n\geq 1 \longrightarrow (4)$

$\exists\, M_2\geq 0$ s.t. $|b_n|\leq M_2 < M_2 + 1\ \forall\, n\geq 1 \longrightarrow (5)$

$\exists\, M_3\geq 0$ s.t. $|c_n|\leq M_3 < M_3 + 1\ \forall\, n\geq 1 \longrightarrow (6)$

If (a_n) and (b_n) are equivalent, then

$\forall\, \varepsilon_4 > 0\ \exists\, N_4\geq 1$ s.t. $|a_n - b_n|\leq\varepsilon_4\ \forall\, n\geq N_4$. $\longrightarrow (7)$.

[All $\varepsilon_1, \dots, \varepsilon_4$ are rationals, $N_1, \dots, N_4$ are positive naturals, $M_1, \dots, M_4$ are naturals.

(a) $(a_n + b_n)$ is a Cauchy sequence.

Fix rational $\varepsilon > 0$. Choose $N = \max\{N_1, N_2\}$

Apply (1) for $\varepsilon_1 = \varepsilon/2$ to get $|a_m - a_n|\leq\varepsilon/2\ \forall\, m,n\geq N_1$

$\Rightarrow |a_m - a_n|\leq\varepsilon/2\ \forall\, m,n\geq N$

Similarly in (2), $\varepsilon_2 = \varepsilon/2$ to get $|b_m - b_n|\leq\varepsilon/2\ \forall\, m,n\geq N$.

Then $|(a_m + b_m) - (a_n + b_n)|\leq|a_m - a_n| + |b_m - b_n|$

$\leq\varepsilon\ \forall\, m,n\geq N$.

$\Rightarrow (a_n + b_n)_{n\geq 1}$ is Cauchy sequence.

(ii) $(a_n b_n)_{n\geq 1}$ is a Cauchy sequence.

Fix rational $\varepsilon > 0$.

Then $|a_m b_m - a_n b_n| = |(a_m - a_n) b_m + a_n(b_m - b_n)|$

$\leq |a_m - a_n|\,|b_m| + |b_m - b_n|\,|a_n|$

$\leq \varepsilon_1(M_2 + 1) + \varepsilon_2(M_1 + 1)\ \forall\, m,n\geq N$.

Now choose $\varepsilon_1 = \dfrac{\varepsilon}{2(M_2 + 1)}$, $\varepsilon_2 = \dfrac{\varepsilon}{2(M_1 + 1)}$

Then $|a_m b_m - a_n b_n|\leq\varepsilon\ \forall\, m,n\geq N$.

(3) If Cauchy sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are equivalent and $(c_n)_{n\geq 1}$ is another Cauchy sequence then $(a_n + c_n)\sim(b_n + c_n)$ and $(a_n c_n)\sim(b_n c_n)$

Proof: (1) Note that $|(a_n + c_n) - (b_n + c_n)| = |a_n - b_n|$

That since $\forall\, \varepsilon_4 > 0\ \exists\, N_4\geq 1$ s.t. $|a_n - b_n|\leq\varepsilon_4\ \forall\, n\geq N_4$, we have $\forall\, \varepsilon_4 > 0\ \exists\, N_4\geq 1$ s.t. $|(a_n + c_n) - (b_n + c_n)|\leq\varepsilon_4\ \forall\, n\geq N_4$.

(2) $|a_n c_n - b_n c_n| = |a_n - b_n|\,|c_n|$

$\leq \varepsilon_1\times(M_3 + 1)\ \forall\, n\geq N_1$

Choosing $\varepsilon_1 = \dfrac{\varepsilon}{M_3 + 1}$, we get

$\forall\, \varepsilon > 0\ \exists\, N_1\geq 0$ s.t. $|a_n c_n - b_n c_n|\leq\varepsilon\ \forall\, n\geq N_1$

§ Lecture 11.1

Thursday, 17 September 2026 18:48

Relationship to rationals: Note that any constant sequence $(a_n)_{n\geq 1}$, where $a_n = a \in \mathbb{Q}$ is Cauchy sequence as $\forall\, \varepsilon > 0$, $\exists N = 1$

$$|a_m - a_n| = 0 < \varepsilon \quad \forall\, m, n \geq N.$$

Then $\lim\limits_{n\to\infty} a = a$ (Then clearly $a+b = \lim\limits_{n\to\infty}(a+b)$

$ab = \lim\limits_{n\to\infty}(ab)$

$a = b \iff \lim\limits_{n\to\infty} a = \lim\limits_{n\to\infty} b$.

$\Rightarrow \mathbb{Q} \subseteq \mathbb{R}$.

For example: $0 = \lim\limits_{n\to\infty} 0$.

Example: Let $(a_n)_{n\geq 1}$, $a_n = \dfrac{2n+5}{3n+2}$. Then $(a_n)_{n\geq 1}$ is a Cauchy sequence and $\lim\limits_{n\to\infty} a_n = 2/3$.

Proof: Consider $\left|a_n - \frac{2}{3}\right| = \left|\dfrac{2n+5}{3n+2} - \dfrac{2}{3}\right|$

$$= \left|\frac{6n+15-6n-4}{3(3n+2)}\right|$$

$$= \frac{11}{9n+6} \leq \frac{11}{9n} \leq \frac{2}{n}.$$

Let $\varepsilon > 0$ be fixed. Then there exists $M > 0$ s.t $0 < \frac{2}{\varepsilon} < M < M+1$.

Choose $N = M+1$, then

$$\left|a_n - \frac{2}{3}\right| \leq \frac{2}{N} \leq \varepsilon \quad \forall\, n \geq N. \quad \to (1)$$

Thus $(a_n)_{n\geq 1}$ and $(2/3)_{n\geq 1}$ are equivalent. So

$$\lim_{n\to\infty}(a_n) = 2/3.$$

Now fix $\tilde{\varepsilon} > 0$.

For $j, k \geq N$, we have $|a_j - a_k| \leq |a_j - 2/3| + |a_k - 2/3|$

$\leq \tilde{\varepsilon}$. where we used $|a_j - 2/3| \leq \tilde{\varepsilon}/2$ $\forall\, j \geq N$.

Thus $(a_n)_{n\geq 1}$ is a Cauchy sequence.

Negation and subtraction: Let $x = \lim\limits_{n\to\infty} a_n$, $y = \lim\limits_{n\to\infty} b_n$ then

$$-x = -1 \times x = \lim_{n\to\infty}(-a_n)$$

$$x - y = x + (-y) = \lim_{n\to\infty}(a_n - b_n).$$

One can now prove: $x(y+z) = xy + yz$ etc.

§ Lecture 11.2

Thursday, 17 September 2026 19:28

Reciprocation: Let $x = \underset{n\to\infty}{\mathrm{LIM}}\,(a_n)$

Then $x^{-1} \overset{?}{=} \underset{n\to\infty}{\mathrm{LIM}}\,(a_n)^{-1}$

This fails even first test of well definedness.

$(a_n) = (10^{-1}, 10^{-2}, 10^{-3}, \cdots)$

$(a_n^{-1}) = (10, 10^2, 10^3, \cdots)$ This is not even bounded "Not Cauchy"

Definition: (Sequences bounded away from zero):

A sequence $(a_n)_{n\geq 1}$ of rationals is said to be bounded away from zero iff $\exists$ rational $c>0$ s.t. $|a_n| > c \;\forall n \geq 1$.

Lemma: Let $x \neq 0$ be a real number. Then there exists a Cauchy sequence $(a_n)_{n\geq 1}$ that is bounded away from zero with $x = \underset{n\to\infty}{\mathrm{LIM}}\, a_n$.

Proof: Since x is real, there is a Cauchy sequence $(b_n)_{n\geq 1}$ of rationals s.t.

$$\underset{n\to\infty}{\mathrm{LIM}}\, b_n = x.$$

Given $x \neq 0 = \underset{n\to\infty}{\mathrm{LIM}}\, 0$; that is sequences $(b_n)_{n\geq 1}$ and $(0)_{n\geq 1}$ are not equivalent. That is

$\exists\, \varepsilon_0 > 0 \;\; \forall N \geq 1 \;\; \exists\, n_0 \geq N$ s.t. $|b_{n_0} - 0| > \varepsilon_0$

Equivalence would mean $\forall \varepsilon > 0 \; \exists N > 0 \; \forall n \geq N \; |a_n - 0| \leq \varepsilon$.

Fix this ε_0. Then there is some $n_0(N) \geq N$ s.t. $|b_{n_0}| > \varepsilon_0$.

$\to$ ① This is true for all N.

But since $(b_n)_{n\geq 1}$ is a Cauchy sequence

$\forall\, \delta > 0 \;\; \exists\, N_0 \geq 1 \;\; \forall\, j, k \geq N_0 \;\; |b_j - b_k| \leq \delta$.

Chose $\delta = \varepsilon_0/2$, and $j = n_0(N_0) \geq N_0$, then

$$|b_{n_0} - b_k| \leq \varepsilon_0/2 \quad \forall\, k \geq N_0.$$

Note that

$$|b_{n_0}| \leq |b_{n_0} - b_k| + |b_k|$$

$$\Rightarrow \quad |b_k| \geq |b_{n_0}| - |b_{n_0} - b_k|$$

$$\geq \varepsilon_0/2 \quad \forall\, k \geq N_0.$$

Now choose

$$a_n = \begin{cases} \varepsilon_0/2 & \forall\, n < N_0 \\ b_n & \forall\, n \geq N_0 \end{cases}$$

Then $|a_n| \geq \varepsilon_0/2 > 0 \quad \forall\, n \geq 1$.

This concludes the proof.

Lemma: Let $(a_n)_{n\geq 1}$ be a Cauchy sequence that is bounded away from zero. Then $(a_n^{-1})_{n\geq 1}$ is also a Cauchy sequence.

Proof: Given: $\exists\, c > 0$ s.t. $|a_n| > c \;\; \forall\, n \geq 1$.

Let $\varepsilon > 0$ be fixed. Note

$$|a_m^{-1} - a_n^{-1}| = \frac{|a_m - a_n|}{|a_m|\,|a_n|}$$

$$\leq c^{-2}\, |a_m - a_n|$$

But $(a_m)_{m\geq 1}$ is a Cauchy sequence then $\exists\, N \geq 1$ s.t.

$$|a_m - a_n| \leq \varepsilon/c^2 \quad \forall\, m, n \geq N. \quad \text{as } \varepsilon/c^2 > 0.$$

Then $|a_m^{-1} - a_n^{-1}| \leq \varepsilon \quad \forall\, m, n \geq N$.

$\Rightarrow$ $(a_m^{-1})_{m\geq 1}$ is a Cauchy sequence.

Reciprocal of real numbers: Let $x \neq 0$. Let $(a_n)_{n\geq 1}$ be a Cauchy sequence that is bounded away from zero s.t. $x = \underset{n\to\infty}{\mathrm{LIM}}\, a_n$. Then

$$x^{-1} := \underset{n\to\infty}{\mathrm{LIM}}\, a_n^{-1}.$$

[Check substitution: If $x = \underset{n\to\infty}{\mathrm{LIM}}\, a_n$, $y = \underset{n\to\infty}{\mathrm{LIM}}\, b_n$, where $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences bounded away from zero such that $x = y$

then $$x^{-1} = \underset{n\to\infty}{\mathrm{LIM}}\, a_n^{-1} = \underset{n\to\infty}{\mathrm{LIM}}\, b_n^{-1} = y^{-1}.$$]