§ Lecture 10.0

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In the beginning we noted that dealing with infinite limits, sums etc require care and underpinning of foundations.

So we started with counting and introduced

1. Naturals via Peano axioms

   They satisfy well ordering principle: Any nonempty subset of natural numbers has a least element.

2. Subtraction needed integers: We built them via algebraic equivalence relation $a-b = c-d \iff a+d = b+c$.

   This destroys well ordering principle.

3. Division needed rationals: We built them via algebraic equivalence relation $a/b = c/d \iff ad = bc$.

   This destroys the notion of "next element".

4. We now showed that there are no numbers of the form $x^2 = 2$ in rationals. But there for each $\varepsilon > 0$ $\exists\, y \in \mathbb{Q}$ s.t.

$$y^2 < 2 < (y+\varepsilon)^2.$$

So there are holes on the line of rationals.

Attempt 1: Take $\mathbb{Q} \cup \{\text{roots of polynomials}\}$

like $x^2 - 2 = 0$ or any other polynomial.

(These are algebraic numbers)

Such a set is countable.

Towards real numbers: We have rational numbers. We know that they approach to something that is not a rational number.

Wherever they approach must be included in the definition.

$$a_0,\ a_1,\ a_2, \dots,\ a_N,\ a_{N+1},\ a_{N+2}, \dots$$

These have to be close.

Sequences: Let $m \in \mathbb{Z}$. A sequence $(a_n)_{n=m}^{\infty}$ of rational numbers is any function from set $\{n \in \mathbb{N} : n \geq m\}$ to $\mathbb{Q}$. That is, a sequence $(a_n)_{n=m}^{\infty}$ is a collection of rationals $a_m, a_{m+1}, \dots$

Cauchy sequences: A sequence $(a_n)_{n\geq 1}$ of rationals is called a Cauchy sequence iff $\forall$ rational $\varepsilon > 0$

$$\exists\, N \geq 1 \text{ s.t. } d(a_j, a_k) \leq \varepsilon \quad \forall\, j,k \geq N.$$

Remark: ① $\forall \varepsilon > 0$ $d(a_j, a_k) \leq \varepsilon \iff \forall \varepsilon > 0$ $d(a_j, a_k) < \varepsilon$.

Proof sketch: ⓐ Strict $\Rightarrow$ non strict. By definition of "<".

ⓑ Nonstrict $\Rightarrow$ strict

$\forall \varepsilon > 0 \quad d(a_j, a_k) \leq \varepsilon \rightarrow ①$

$\Rightarrow$ Fix any $\varepsilon' > 0$. Then from ① $d(a_j, a_k) \leq \varepsilon'/2$

$\Rightarrow d(a_j, a_k) < \varepsilon'$.

② ε is a rational. We need to mind this until reals are introduced.

Example: The sequence $(a_n)_{n\geq 1}$, where $a_n = \frac{1}{n}$ is a Cauchy sequence.

Solution: We need to show that $\forall$ rational $\varepsilon > 0$, $\exists N \geq 1$ s.t.

$$d(a_j, a_k) \leq \varepsilon \quad \forall\, j,k \geq N.$$

What does it mean? We need to find N s.t. above holds for all $\varepsilon > 0$.

Fix $\varepsilon > 0$. Then $d(a_j, a_k) = |a_j - a_k| = \left|\frac{1}{j} - \frac{1}{k}\right|$

Since $j,k \geq N \Rightarrow \frac{1}{j}, \frac{1}{k} \leq \frac{1}{N}$. Also $\frac{1}{j}, \frac{1}{k} > 0$. Then

$$-\frac{1}{N} \leq \frac{1}{j} - \frac{1}{k} \leq \frac{1}{N} \quad \text{or} \quad \left|\frac{1}{j} - \frac{1}{k}\right| \leq \frac{1}{N}.$$

Note that from Archimedean property of rationals, $\exists$ an integer N s.t.

$$N - 1 \leq 1/\varepsilon < N \Rightarrow \frac{1}{N} < \varepsilon$$

Thus $d(a_j, a_k) \leq \varepsilon$.

This completes the proof.

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Bounded sequences. A sequence $(a_n)_{n\geq 1}$ of rationals is said to be bounded iff there exists some rational $M \geq 0$ s.t. $|a_i| \leq M \; \forall \; i \geq 1$.

Finite sequences: A sequence $(a_n)_{n=1}^{M}$ is called a finite sequence, where $M \in \mathbb{N}$. Because the set $\{n : 1 \leq n \leq M\}$ is finite with cardinality M.

On the other hand $(a_n)_{n\in\mathbb{N}}$ is infinite.

Claim: Every finite set is bounded. That is for all $M \in \mathbb{N}$ $(a_n)_{n=1}^{M}$ is a bounded sequence.

Proof: Apply induction on M.

Base case: $M=1$. The sequence is a_1, which is bounded by $|a_1|$.

Inductive step: Assume $(a_n)_{n=1}^{M}$ is bounded by M_0. i.e.

$$\forall \; 1 \leq n \leq M \qquad |a_n| \leq M_0.$$

Then $(a_n)_{n=1}^{M+1}$ is bounded by M_0', where

$$M_0' = \max\{M_0, |a_{M+1}|\}.$$

It is clear that $|a_n| \leq M_0' \quad \forall \; 1 \leq n \leq M+1$.

Lemma: Every Cauchy sequence $(a_n)_{n\geq 1}$ is bounded.

Proof: Let $(a_n)_{n\geq 1}$ be a Cauchy sequence. Then $\forall \; \varepsilon > 0$, $\exists \; N \geq 1$ s.t.

$\forall \; j, k \geq N \qquad |a_j - a_k| \leq \varepsilon.$

Choose $\varepsilon = 1$, $k = N$, then $\forall \; j \geq N$

$$|a_j - a_N| \leq 1. \longrightarrow (1)$$

Then $|a_j| = |a_j - a_N + a_N| \leq |a_j - a_N| + |a_N|$

$$\leq 1 + |a_N|. \longrightarrow (2)$$

The sequence $a_1, \ldots, a_{N-1}$ is finite and hence bounded by some $M_N \geq 0$, i.e. $|a_i| \leq M_N \; \forall \; i \leq N-1$.

Then $|a_i| \leq M' \quad \forall \quad i \geq 1$, where

$$M' = \max\{M_N, 1 + |a_N|\}.$$

Definition (Equivalent sequences). Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of rational numbers. Then we say that $(a_n)_{n\geq 1}$ is equivalent to $(b_n)_{n\geq 1}$ iff $\forall$ rational $\varepsilon > 0$, $\exists \; N \geq 1$ s.t. $|a_n - b_n| \leq \varepsilon \; \forall \; n \geq N$.

This can be extended to positive reals when we define them.

Example: Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences s.t. $a_n = 1 + 10^{-n}$ and $b_n = 1 - 10^{-n}$. Then (a_n) and (b_n) are equivalent.

Solution: Fix some $\varepsilon > 0$.

We need to find $N \geq 1$ s.t. $|a_n - b_n| < \varepsilon \; \forall \; n \geq N$.

Note that $|a_n - b_n| = 2 \times 10^{-n}$

$$\leq 2 \times 10^{-N} \qquad \forall \; n \geq N.$$

We need to prove that $\exists \; N$ s.t. $2 \times 10^{-N} \leq \varepsilon$.

How? Note that $10^N \geq N \quad \forall \; N \geq 1$ (Induction)

$$\Rightarrow \quad 2 \times 10^{-N} \leq \frac{2}{N}.$$

From archimedian property $\exists$ integer N s.t. $N - 1 \leq \frac{2}{\varepsilon} < N$, i.e. $\frac{2}{N} < \varepsilon$.

$\Rightarrow 2 \times 10^{-N} < \varepsilon$ or $|a_n - b_n| \leq \varepsilon \; \forall \; n \geq N$. [Since $2/\varepsilon > 0$, $N \geq 1$.]

Real numbers: A real number is defined to be an object of the form $\mathrm{LIM}_{n\to\infty} a_n$, where $(a_n)_{n\geq 1}$ is a Cauchy sequence of rationals. Two real numbers $\mathrm{LIM}_{n\to\infty} a_n$ and $\mathrm{LIM}_{n\to\infty} b_n$ are said to be same iff $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are equivalent Cauchy sequences.

[$\mathrm{LIM}_{n\to\infty} a_n$ is only formal notation.]

Remark: The equality is well defined. Let

$x = \mathrm{LIM}_{n\to\infty} a_n$, $y = \mathrm{LIM}_{n\to\infty} b_n$, $z = \mathrm{LIM}_{n\to\infty} c_n$ then

(i) $x = x$ (ii) $x = y \Leftrightarrow y = x$ (iii) $x = y$ & $y = z \Rightarrow x = z$.

Arithmatic operations on reals.

Addition and multiplication:- Let $x = \mathrm{LIM}_{n\to\infty} a_n$ and $y = \mathrm{LIM}_{n\to\infty} b_n$ be two real numbers then

$$x + y := \mathrm{LIM}_{n\to\infty} (a_n + b_n)$$

$$xy := \mathrm{LIM}_{n\to\infty} (a_n b_n)$$

Requirements:

(1) If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are two Cauchy sequences of rationals then $(a_n + b_n)_{n\geq 1}$ and $(a_n b_n)_{n\geq 1}$ are Cauchy sequences themselves.

Proof: