Course information :

1. Book : Analysis I by Terence Tao
2. Contact : uttam @ iiit.ac.in
3. Course material will be uploaded at uttamcft.github.io/MA4.101-Real-Analysis-M26

Q. What is real analysis:

It is an analysis of

1. Real numbers
2. sequences and series of real numbers
3. Functions of real numbers

There are other types as well:

→ Complex analysis

→ Functional analysis

→ Harmonic analysis

⋮

Aside

But we all know "at some level" these concepts and we can compute things like limits, summation of series, derivatives and integrals.

⇒ Here we focus on foundations of these objects, e.g., why you compute limits like you do ?

⇒ Things that we will be interested in

(a) What is a real number ?

→ What is the largest real number ?

→ After 0, what is the next real number.

→ Why 2 has square root but -2 doesn't ?

→ How many real numbers are there ?

→ Are there more real numbers than rational numbers ?

(b) Which sequences have limit and which don't and why ?

→ Can you add infinitely many real numbers and still get a finite value ?

→ Can you add infinitely many rational numbers and get non rational ?

(c) What is a function ?

→ what continuity means ?

→ Can one differentiate an infinite series of functions ?

How things work ? e.g. $\int_{\pi/2}^{0} \cos x \, dx = 1$

Why things work ?

But since you mostly know how to compute why bother ?

↓

One can get in trouble if one doesn't know why !!

1. $ac = bc \overset{?}{\Rightarrow} a = b$

2. (a) $S = 1 + \frac{1}{2} + \frac{1}{4} + \cdots$

$2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$

$2S = 2 + S \Rightarrow S = 2$

(b) $S = 1 + 2 + 4 + \cdots$

$2S = 2 + 4 + 8 + \cdots = S - 1$

$\Rightarrow S = -1$

So which computation should you trust ?

(c) $S = 1 - 1 + 1 - 1 + \cdots$

$= 1 - (1 - 1 + 1 - \cdots) = 1 - S$

$\Rightarrow S = 1/2$

(d) $S = (1-1) + (1-1) + \cdots$

$= 0 + 0 + \cdots = 0$

(e) $S = 1 + (-1+1) + (-1+1) + \cdots$

$= 1$

Three different answers.

3. Let $x \in \mathbb{R}$,

$$L = \lim_{n\to\infty} x^n$$

$$= \lim_{m+1\to\infty} x^{m+1}$$

$$= x \lim_{m\to\infty} x^m = xL$$

$\Rightarrow L(1-x) = 0 \Rightarrow x = 1$ or $L = 0$

that is $\forall x \neq 1, \lim_{n\to\infty} x^n = 0.$

Take $x = -1$; $1, -1, 1, -1 \cdots$ sequence converges to zero !!

This can't be true, what is wrong ??

4. $$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij}$$

But $$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} \overset{?}{=} \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

5. $$\lim_{x\to 0} \lim_{y\to 0} \frac{x^2}{x^2+y^2} \overset{?}{=} \lim_{y\to 0} \lim_{x\to 0} \frac{x^2}{x^2+y^2}$$

LHS $= \lim_{x\to 0} 1 = 1$

$\neq$ Not equal, why ?

RHS $= \lim_{y\to 0} \frac{0}{0+y^2} = 0$

§ Lecture 1.1

Monday, 10 August 2026   19:49

The Beginning :   What are numbers?

1. Natural numbers
2. Rational numbers
3. Real numbers
4. complex numbers
5. Quaternians

None seem to be special !!

So again what are numbers?

$$\text{Naturals} \longrightarrow \text{Intigers} \longrightarrow \text{Rationals} \longrightarrow \text{Reals}$$

You know how to use! (Knowing a computer)

We care how to define? (Building a computer)

We will not use anything that we didn't define.

The Peano axioms (1858–1932) :

This is one approach [of course the numbers were known much much before]

Infamal

$N := \{0, 1, 2, \cdots\}$

① N is the set of natural numbers (# we didn't define set)

② Any element of N is a natural number (# we didn't define what it means by in N).

Start with zero and keep counting indefinitely.

Q. But how do we know we can do that? may be we come back to zero?

Q. How do we define addition, multiplication etc?

Insight : Exponentiation $5^3 = 5 \times 5 \times 5$

Multiplication $5 \times 3 = 5 + 5 + 5$

Addition hmm !!

$5+3$ = start with three perform increment of unit three times.

increment ??

Define
0
$0{+}{+}$
$(0{+}{+}){+}{+}$
$\vdots$

Axiom 1 : 0 is a natural number.

Axiom 2 If n is a natural number then $n{+}{+}$ is also a natural number.

Convention
0
$1 := 0{+}{+}$
$2 := (0{+}{+}){+}{+} = 1{+}{+}$
$\vdots$

Above two doesn't forbid $\{0,1\} = B$

0
$0{+}{+} = 1$
$1{+}{+} = 0$

Axiom 3. 0 is not the successor of any natural number, i.e.

$n{+}{+} \neq 0$ for any natural number n.

e.g. $4 \neq 0$ as $3{+}{+}$ can't be zero.

Other pathology :

$$\left.\begin{array}{r} 0{+}{+} = 1 \\ 1{+}{+} = 2 \\ 3 := 2{+}{+} = 2 \end{array}\right\}$$

Axiom 4 Different natural numbers must have different successors, i.e.

For natural numbers n, m

$n \neq m \Rightarrow n{+}{+} \neq m{+}{+}$

$\Updownarrow$ [Contrapositive]

$n{+}{+} = m{+}{+} \Rightarrow n = m$

Are we done? $0, 1, 2, 3 \ldots$ are all distinct now.

Consider

$N' = \{0, 0.5, 1, 1.5, 2, \cdots\}$

① $0 \in N'$ ② $0{+}{+} = 1 \in N'$ and so on.

③ 0 is not successor any number.

④ if m, n s.t. $m \neq n \Rightarrow m{+}{+} \neq n{+}{+}$.

Hmm.

What is not true is that each number is not reachable from 0 via increment operation.

Axiom 5: (Principle of mathematical induction).

Let $P(n)$ be any property pertaining to a natural number n. Suppose

① $P(0)$ is true.

② $P(n)$ is true $\Rightarrow P(n{+}{+})$ is true.

Then $P(n)$ is true for all natural numbers.

e.g. $P(n)$ could be "n is even".

" "$n+1$ is odd"

" "n solves $n^2 = 4$"

$\vdots$

"These are infinitely many axioms"

Suppose $P(n)$ = "n is not a half integer"

$P(0)$ = 0 is not a half integer is true.

Suppose $P(n)$ is true, i.e. n is not a half integer then

$\Rightarrow n{+}{+}$ " "

$\Rightarrow P(n{+}{+})$ is true

$\Rightarrow P(n)$ is true for all natural numbers.

Recipe :

Claim: A certain property $P(n)$ is true for all natural numbers.

Proof: ① show that $P(0)$ is true.

② Assume $P(n)$ is true then show that $P(n{+}{+})$ is true as well.

Apply axiom 5 to conclude that $P(n)$ is true for all natural numbers.

Assumption: A number system satisfying Peano axioms is called a set of natural numbers. (Denoted as N).

$0, 1, 2, 3, \cdots$

$0, \mathrm{I}, \mathrm{II}, \mathrm{III}, \cdots$

Different representations

$0 \Leftrightarrow 0$
$1 \Leftrightarrow \mathrm{I}$
$\vdots$

We didn't say what the natural numbers are. We gave some axioms (didn't say what they measure, count etc)