Cauchy sequences and their equivalence

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Claim 1: Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two Cauchy sequences of rationals. Then

(i) $(a_n+b_n)_{n\geq 1}$ is a Cauchy sequence.

(ii) $(a_n b_n)_{n\geq 1}$ is a Cauchy sequence.

Proof. Note the following points.

Since $(a_n)_{n\geq 1}$ is a Cauchy sequence, we have

$$\boxed{\forall \text{ rational } \delta_1 > 0 \quad \exists\, N_1 \geq 1 \text{ s.t. } |a_m - a_n| \leq \delta_1 \quad \forall\, m,n \geq N_1} \longrightarrow (1)$$

Since $(b_n)_{n\geq 1}$ is a Cauchy sequence, we have

$$\boxed{\forall \text{ rational } \delta_2 > 0 \quad \exists\, N_2 \geq 1 \text{ s.t. } |b_m - b_n| \leq \delta_2 \quad \forall\, m,n \geq N_2} \longrightarrow (2)$$

Since $(a_n)_{n\geq 1}$ is a Cauchy sequence, it is bounded. That is

$$\boxed{\exists \text{ rational } M_1 \geq 0 \text{ s.t. } |a_n| \leq M_1 < M_1+1 \quad \forall\, n\geq 1} \longrightarrow (3)$$

Since $(b_n)_{n\geq 1}$ is a Cauchy sequence, it is bounded. That is

$$\boxed{\exists \text{ rational } M_2 \geq 0 \text{ s.t. } |b_n| \leq M_2 < M_2+1 \quad \forall\, n\geq 1} \longrightarrow (4)$$

[This "rational" quantifier will be dropped later].

(i) Fix rational $\varepsilon > 0$.

From eqn (1) and (2) two naturals N_1, N_2 exist. Let

$$N := \max\{N_1, N_2\}.$$

From (1) we get $\quad |a_m - a_n| \leq \varepsilon/2 \quad \forall\, m,n \geq N_1 \quad$ (Choose $\delta_1 = \varepsilon/2 > 0$)

since $N \geq N_1$, $\quad |a_m - a_n| \leq \varepsilon/2 \quad \forall\, m,n \geq N$.

Similarly (2) gives $\quad |b_m - b_n| \leq \varepsilon/2 \quad \forall\, m,n \geq N_2 \quad$ (Choose $\delta_2 = \varepsilon/2 > 0$)

Since $N \geq N_2$, $\quad |b_m - b_n| \leq \varepsilon/2 \quad \forall\, m,n \geq N$.

Then $\quad |a_m + b_m - a_n - b_n| \leq |a_m - a_n| + |b_m - b_n|$

$\qquad \leq \varepsilon \quad \forall\, m,n \geq N$, where

N exists and equal to $\max\{N_1, N_2\}$. Further, ε was arbitrary so above holds for all $\varepsilon > 0$.

$\Rightarrow$ $(a_n+b_n)_{n\geq 1}$ is a Cauchy sequence.

(ii) Fix $\varepsilon > 0$. Then

$$|a_m b_m - a_n b_n| = |a_m b_m - a_n b_m + a_n b_m - a_n b_n|$$

$$\leq |a_m - a_n|\,|b_m| + |a_n|\,|b_m - b_n|$$

$$\leq \delta_1 (M_2+1) + \delta_2 (M_1+1)$$

[Using eqn (1) – (4)] $\quad \forall\, m,n \geq N = \max\{N_1, N_2\}$

Since above is true for all $\delta_1 > 0$ & $\delta_2 > 0$, choose

$$\underbrace{\delta_1 = \frac{\varepsilon}{2(M_2+1)}}_{\text{Rational}} \quad \& \quad \underbrace{\delta_2 = \frac{\varepsilon}{2(M_1+1)}}_{\text{Rational}}, \quad \text{to obtain}$$

(Hence we can choose)

$$|a_m b_m - a_n b_n| \leq \varepsilon \quad \forall\, m,n \geq N.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $(a_n b_n)_{n\geq 1}$ is a Cauchy sequence.

Claim 2: Let two Cauchy sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ of rationals be equivalent. Let $(c_n)_{n\geq 1}$ be another Cauchy sequence of rationals, then

(i) $(a_n + c_n)_{n\geq 1}$ is equivalent to $(b_n + c_n)_{n\geq 1}$.

(ii) $(a_n c_n)_{n\geq 1}$ is equivalent to $(b_n c_n)_{n\geq 1}$.

Proof: Since $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are Cauchy sequences, we have equations (1) → (4). Further, $(c_n)_{n\geq 1}$ is a Cauchy sequence, we have

$$\boxed{\forall \text{ rational } \delta_3 > 0 \quad \exists\, N_3 \geq 1 \text{ s.t. } |c_m - c_n| \leq \delta_3 \quad \forall\, m,n \geq N_3} \longrightarrow (5)$$

$$\boxed{\exists \text{ rational } M_3 \geq 0 \text{ s.t. } |c_n| \leq M_3 < M_3+1 \quad \forall\, n\geq 1} \longrightarrow (6)$$

Further since $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are equivalent, we have

$$\boxed{\forall \text{ rational } \delta_4 > 0 \quad \exists\, N_4 \geq 1 \text{ s.t. } |a_n - b_n| \leq \delta_4 \quad \forall\, n \geq N_4} \longrightarrow (7)$$

(i) Fix $\varepsilon > 0$. Note that

$$|a_n + c_n - (b_n + c_n)| = |a_n - b_n|$$

Using (7) for $\delta_4 = \varepsilon$, we obtain

$$|a_n + c_n - (b_n + c_n)| = |a_n - b_n| \leq \varepsilon \quad \forall\, n \geq N_4.$$

$\Rightarrow$ $(a_n + c_n)_{n\geq 1}$ is equivalent to $(b_n + c_n)_{n\geq 1}$.

(ii) Fix $\varepsilon > 0$. Note that

$$|a_n c_n - b_n c_n| = |a_n - b_n|\,|c_n| \qquad [\text{Using (6) \& (7)}]$$

$$\leq \delta_4 \times (M_3+1) \quad \forall\, n \geq N_4$$

Choosing $\quad \delta_4 = \dfrac{\varepsilon}{M_3+1} \quad$ (rational, and hence possible),

we get

$$|a_n c_n - b_n c_n| \leq \varepsilon \quad \forall\, n \geq N_4.$$

$\Rightarrow$ $(a_n c_n)_{n\geq 1}$ and $(b_n c_n)_{n\geq 1}$ are equivalent.