# § Additional: Cardinal Arithmatic

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Cardinal arithmatic: Let $X, Y$ be finite sets and $Z$ be any set. Then

(a) If $x \notin X$, then $|X \cup \{x\}| = |X| + 1$.

(b) $|X \cup Y| \leq |X| + |Y|$ and equality holds if $X \cap Y = \phi$.

(c) If $Y \subseteq X$, then $|Y| \leq |X|$.

(d) Let $f: X \to Z$ be a function. Then $|f(X)| \leq |X|$. Further if $f$ is injective then $|f(X)| = |X|$.

(e) $|X \times Y| = |X| \times |Y|$

(f) $|Y^X| = |Y|^{|X|}$

Proof: Each has an inductive proof except (a).

(a). Since $X$ is finite, let $|X| = n$. That is there is a bijective function $f: X \to [n]$.

Let $x \notin X$, define another function

$g: X \cup \{x\} \to [n_{++}]$ by

$g(y) = f(y) \quad \forall y \in X$

$g(x) = n_{++}$.

Claim: $g: X \cup \{x\} \to [n_{++}]$ is a bijective function.

① Injectivity: $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$.

Case 1: If $y_1, y_2 \in X$ then

$g(y_1) = g(y_2) \Rightarrow f(y_1) = f(y_2) \Rightarrow y_1 = y_2$

Case 2 If $y_1, y_2 \in \{x\}$ then $y_1 = y_2 = x$

Nothing to prove.

Case 3: If $y_1 \in X$ and $y_2 \in \{x\}$.

$g(y_1) = g(y_2) \Rightarrow f(y_1) = g(x)$

$\Rightarrow f(y_1) = n_{++} \Rightarrow n_{++} \leq n$

$\Rightarrow n < n$

contradiction.

Case 4: Same as case 3.

Surjectivity: For each $m \in [n_{++}]$, there is some $z \in X \cup \{x\}$ s.t. $g(z) = m$.

Case 1: $m \leq n$, then because of surjectivity of $f$, we can find some $z \in X$ s.t. $f(z) = m$

$\Rightarrow g(z) = m$ as $z \in X$.

Case 2: $m = n_{++}$, then we have $z = x$ s.t. $g(x) = n_{++}$.

In both cases there is $z \in X \cup \{x\}$.

$\Rightarrow \boxed{|X \cup \{x\}| = |X| + 1}$

(b) $|X \cup Y| \leq |X| + |Y|$ and equality holds when $X \cap Y = \phi$.

Proof: Define property

$P(n) :=$ " For finite sets $X$ and $Y$ with $|Y| = n$, $|X \cup Y| \leq |X| + n$ and equality holds when $X \cap Y = \phi$ ".

Base case: $P(0) \to$ Since $|Y| = 0 \Rightarrow Y = \phi$. Then $X \cup Y = X \cup \phi = X$.

And $|X \cup Y| = |X| + 0$ ( Equality holds because $X \cap \phi = \phi$).

Thus $P(0)$ is true.

Inductive case: Assume $P(n)$ is true. Let $|Y| = n_{++}$. Then let $y \in Y$.

Let $Y' = Y - \{y\}$. Note that $X \cup Y = (X \cup Y') \cup \{y\}$. $|Y'| = n$.

Case 1: $y \in X$. Then $|X \cup Y| = |(X \cup Y') \cup \{y\}| = |X \cup Y'| \leq |X| + n \leq |X| + n_{++}$.

Case 2: $y \notin X$. Then $|X \cup Y| = |X \cup Y' \cup \{y\}| = |X \cup Y'| + 1$

$\leq |X| + n + 1$.

Now let $X \cap Y = \phi$. Then since $y \in Y$, $y \notin X$.

Since $y \notin Y'$ and $y \notin X$, we have

$|X \cup Y| = |X \cup Y' \cup \{y\}| = |X \cup Y'| + 1$

$= |X| + |Y'| + 1$ as $X \cap Y' = \phi$ $[Y' \subseteq Y]$.

This completes the proof.

(c) Let $Y \subseteq X$, then $|Y| \leq |X|$.

Proof: $P(n) =$ " For every $X$ with $|X| = n$, and every $Y \subseteq X$, $|Y| \leq n$. "

Base case: $P(0)$ ; $|X| = 0 \Rightarrow X = \phi \Rightarrow Y = \phi$ Then $|Y| = 0 = |X|$.

Inductive step: Assume $P(n)$ is true. Let $|X| = n_{++}$.

Let $x \in X$, $X' = X - \{x\}$. Then $|X'| = n$. Let $Y \subseteq X$.

Case 1: $x \in Y$. Then $Y - \{x\} \subseteq X'$ as $Y \subseteq X$.

$\Rightarrow |Y - \{x\}| \leq n$.

Since $Y = (Y - \{x\}) \cup \{x\} \Rightarrow |Y| = |Y - \{x\}| + 1$

$\leq n_{++}$

Case 2. $x \notin Y$. Since $Y \subseteq X \Rightarrow Y \subseteq X'$

$\Rightarrow |Y| \leq n \leq n_{++}$.

This concludes the proof.

(d) Let $X$ be finite and $Z$ any set and $f: X \to Z$ any function. Then $|f(X)| \leq |X|$ and equality holds if $f$ is injective.

Proof: $P(n) =$ " For all $X$ with $|X| = n$ and any function $f: X \to Z$ $|f(X)| \leq |X|$ and equality holds if $f$ is injective. "

Base case: $P(0)$: $|X| = 0 \Rightarrow X = \phi$. Then $f: \phi \to Z$ is a unique injective map. Then $f(X) = \phi \Rightarrow |f(X)| = 0 = |X|$.

Inductive step: Assume $P(n)$ is true. Let $|X| = n_{++}$. Let $x \in X$. and $X' = X - \{x\}$.

Then $|X'| = n$. Let $f': X' \to Z$ and $f \in X \to Z$. be the map s.t.

$f(y) = f'(y) \ \forall y \in X'$. Then $f(X) = f'(X') \cup \{f(x)\}$.

$|f'(X')| \leq |X'| = n$.

Case 1: $f(x) \in f'(X') \Rightarrow f(X) = f'(X')$ and $P(n)$ gives

$|f(X)| = |f'(X')| \leq |X'| = n \leq n_{++}$.

Case 2. $f(x) \notin f'(X') \Rightarrow |f(X)| = |f'(X')| + 1 \leq n_{++}$.

For injective $f$, case 1 is impossible. as it amounts to having $x \neq x'$ s.t. $f(x) = f(x')$. The case two is already equality.

(e) $|X \times Y| = |X| \times |Y|$.

Proof: $P(n) =$ " For all finite $X$ and $Y$ with $|Y| = n$, $|X \times Y| = n|X|$ ".

Base case $P(0)$. $\Rightarrow Y = \phi$. Then $X \times Y = X \times \phi = \phi$.

$|X \times Y| = 0 = 0 \cdot |X|$.

Inductive step: Let $P(n)$ be true. Let $|Y| = n_{++}$. Let $y \in Y$. Let $Y' = Y - \{y\}$. Then $|X \times Y'| = n|X|$.

Note that $X \times Y = X \times (Y' \cup \{y\})$

$= (X \times Y') \cup (X \times \{y\})$

$X \times Y' \cap X \times \{y\} = \phi \Rightarrow |X \times Y| = |X \times Y'| + |X \times \{y\}|$

$= n|X| + |X|$

$= n_{++}|X|$.

Here we used $|X \times \{y\}| = |X|$.

This can be argued by noting that $g: X \to X \times \{y\}$ is injective

So $|g(X)| = |X| \Rightarrow |X \times \{y\}| = |X|$. [using (d)].

(f) $|Y^X| = |Y|^{|X|}$.

Proof: $P(n) =$ " For all $X$ with $|X| = n$, and finite $Y$, $|Y|^{|X|} = |Y^X|$. "

Base case: $n = 0$, $|X| = 0 \Rightarrow X = \phi$.

$Y^X = Y^\phi = \{f : f: \phi \to Y\} = \{f\}$

$\Rightarrow |Y^X| = 1 = |Y|^{|\phi|} = |Y|^{|X|}$.

Inductive step: Let $P(n)$ be true. Let $|X| = n_{++}$, Let $X' = X - \{x\}$ for some $x \in X$.

Then define $\Phi: Y^X \to Y^{X'} \times Y$ as $\Phi(f) = (f|_{X'}, f(x))$

$\Phi$ is a bijection. Then $|Y^X| = |Y^{X'} \times Y|$

$= |Y^{X'}| \, |Y|$

$= |Y|^{|X'|} \cdot |Y| = |Y|^{n_{++}}$.

Claim: $\Phi: Y^X \to Y^{X'} \times Y$ with $\Phi(\underset{f: X \to Y.}{f}) = (f|_{X'}, f(x))$ is a bijection.

Proof: Injective: Let $\Phi(f) = \Phi(g)$

$f|_{X'} = g|_{X'}$ and $f(x) = g(x)$. Then $f, g$ agree at every point of $X = X' \cup \{x\}$. Then $f = g$.

Surjective: Given $(h, y)$, define $f: X \to Y$ by $f(z) = h(z) \ \forall z \in X'$ and $f(x) = y$. Then $\Phi(f) = (h, y)$.