§ Additional proofs

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Lemma 1: Suppose that $n \geq 1$, and X has cardinality n. Then X is nonempty, and if $x \in X$, then $X - \{x\}$ has cardinality m such that $m_{++} = n$.

Proof: X is nonempty: This is because there cannot be a bijective mapping from the null set to $[n] = \{i : 1 \leq i \leq n\}$. (This is because any element in $[n]$ has no preimage in ϕ).

Now let $x \in X$. Since X has cardinality n, there is a bijective map $f : X \to [n]$. Then $f(x) \in [n]$.

Define $g : X - \{x\} \longrightarrow [m]$ such that $m_{++} = n$. by the rule

$$g(y) = \begin{cases} f(y) & \text{if } f(y) < f(x) \\ f'(y) & \text{if } f(y) > f(x) \end{cases} \qquad \forall y \in X - \{x\}$$

where $(f'(y))_{++} = f(y)$.

Note that since $y \neq x$, $f(y) \neq f(x)$.

We need to prove that g is a bijective map.

Injectivity: $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$.

Four cases:

① $f(y_1) < f(x)$, $f(y_2) < f(x)$

Then $g(y_1) = g(y_2) \implies f(y_1) = f(y_2) \Rightarrow y_1 = y_2$

② $f(y_1) > f(x)$, $f(y_2) > f(x)$

$g(y_1) = g(y_2) \Rightarrow g(y_1)_{++} = g(y_2)_{++} \Rightarrow f(y_1) = f(y_2) \Rightarrow y_1 = y_2$.

③ $f(y_1) < f(x)$ and $f(y_2) > f(x)$, ie.

$$f(y_1) < f(x) < f(y_2) \Leftrightarrow g(y_1) < f(x) < g(y_2)_{++}$$

$$\Leftrightarrow g(y_1)_{++} \leq f(x) < g(y_2)_{++}$$

But $g(y_1) = g(y_2) = l$ then we have $l_{++} \leq f(x) < l_{++}$

$\Rightarrow l_{++} < l_{++}$ [Contradiction]

④ $f(y_2) < f(x) < f(y_1)$ (Same contradiction as above).

Thus, g is an injective function.

Surjectivity: Let $k \in [m] = \{i : 1 \leq i \leq m\}$, $m_{++} = n$.

Case 1: $k < f(x)$.

Then $1 \leq k \leq m < n \Rightarrow k \in [n]$ and since f is surjective there is a $y \in X$ s.t. $k = f(y)$.

Now $f(y) = k < f(x) \Rightarrow f(y) \neq f(x). \Rightarrow y \neq x$

$\Rightarrow y \in X - \{x\}$.

Also since $f(y) < f(x) \Rightarrow g(y) = f(y) = k$.

Case 2: $k \geq f(x)$

$k \leq m \Rightarrow k_{++} \leq m_{++} = n$

Then there is a $y \in X$ s.t. $f(y) = k_{++}$.

$k \geq f(x) \Rightarrow k_{++} > f(x)$

or $f(y) > f(x) \Rightarrow y \neq x$ So $y \in X - \{x\}$.

And $f(y) = g(y)_{++} \Leftrightarrow k_{++} = g(y)_{++} \Rightarrow k = g(y)$.

Thus g is surjective.

Thus, $X - \{x\}$ has cardinality m with $m_{++} = n$.

Lemma 2 (Uniqueness of cardinality): Let X be a set with cardinality n. Then X can't have cardinality $m \neq n$.

Proof: We do induction on n.

Base case: We need to show that if a set has cardinality 0 then it can't have non zero cardinality.

→ If the cardinality is zero then X must be empty. Now empty set cannot be mapped to nonempty set. Thus, lemma is proven for $n = 0$.

Inductive step: Suppose the lemma is proven for some n. We need to prove the lemma for n_{++}.

Suppose X has cardinality n_{++}. Suppose it has cardinality $p_{++} \neq n_{++}$ as well.

Since X is nonempty, for any $x \in X$, $X - \{x\}$ has cardinality p and n.

But from inductive hypothesis $p = n$.

$\Rightarrow n_{++} = p_{++}$

That is lemma is true for n_{++} and hence for all $n \in N$.