International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2026)

Quiz 1

Time: 45 Minutes | Total Marks: 15

Guidelines

1. The algebra of natural numbers is allowed but one cannot use subtraction, division, etc.
2. The axioms used to solve any question must be stated at least once in the solution.
3. The use of any axiom must be demonstrated clearly. Failing this will result in deduction of marks. For example, saying that "Axiom 5 yields this result" is a meaningless statement unless it is shown as to how is axiom 5 invoked.
4. These facts may not be proved in your solution: For $a, b, c \in \mathbb{N}$ (1) $a < b$ implies $ac < bc$ for $c > 0$. (2) $(a+b)^2 = a^2 + b^2 + 2ab$. (3) $a < b$ iff $a{+}{+} \leq b$. (4) for $c > 0$, $ac = bc$ implies $a = b$.
5. If your total across all parts is $\boldsymbol{s}$, the mark awarded is $\boldsymbol{s \ominus (s \ominus 15)}$, where $\ominus$ is the truncated subtraction of Question 1 of Assignment 1.

Question 1 [6 Marks]. Prove that $n^2 \leq 2^n$ for every $n \geq 4$. Determine every $n \in \mathbb{N}$ for which it holds.

Question 2 [9 Marks]. Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that $f(0) = 0$ and

$$f(a) + g(a) + 1 \;\leq\; f(a+1) \qquad \text{for all } a \in \mathbb{N}, \tag{1}$$

and set $S_{f,g} := \{\, f(a) + b \;:\; a, b \in \mathbb{N},\; b \leq g(a) \,\}$. In parts (a), (b) and (c) you may use the following two consequences of Eq. (1) *without proof.*

(P1) f is strictly increasing: $f(a) < f(c)$ whenever $a < c$.

(P2) $f(a) + g(a) + 1 \leq f(c)$ whenever $a < c$.

Prove the following.

(a) [**2 Marks**] The representation is unique: if $f(a) + b = f(c) + d$ with $b \leq g(a)$ and $d \leq g(c)$, then $a = c$ and $b = d$.

(b) [**4 Marks**] If $f(a+1) = f(a) + g(a) + 1$ for every $a \in \mathbb{N}$, then $S_{f,g} = \mathbb{N}$.

(c) [**3 Marks**] If $f(a+1) \neq f(a) + g(a) + 1$ for some $a \in \mathbb{N}$, then $S_{f,g} \neq \mathbb{N}$.

(d) [**Bonus, 2 Marks**] Prove (P1) and (P2) from Eq. (1). Attempt this only after completing (a), (b) and (c).