International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2026)

Practice Problems 2

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Disclaimer

Please note the following.

1. The solutions given here are not the only ones possible.
2. Use the solutions as guidelines.
3. If you find any mistakes or typos, you may reach out and it shall be cleared.
4. If something does not make sense in solutions, you may reach out and we shall clarify them in tutorials and/or classes.
5. Axioms are numbered as in class: Axiom k is Tao's Axiom $3.k$ for set theory and Axiom $2.k$ for the Peano axioms.
6. For a function $f : X \to Y$ we continue to call X the *domain* and Y the *range*, as in class.

A useful Lemma

Lemma P (componentwise membership). *Let p be an ordered pair, say $p = (x, y)$. Then for all sets X and Y,*

$$p \in X \times Y \quad \Longleftrightarrow \quad (x \in X \text{ and } y \in Y).$$

Moreover the components x and y are determined by p.

Proof. By definition, $p \in X \times Y$ holds if and only if $p = (x', y')$ for some $x' \in X$ and $y' \in Y$. If $x \in X$ and $y \in Y$ we may take $x' := x$ and $y' := y$. Conversely, if $p = (x', y')$ with $x' \in X$, $y' \in Y$, then $(x, y) = (x', y')$, so $x = x'$ and $y = y'$; hence $x \in X$ and $y \in Y$. The same equation, $(x, y) = (x', y') \Leftrightarrow (x = x' \text{ and } y = y')$, gives that the components are determined by p.

Question (1) [Exercise 3.5.4 of Tao].

Let A, B, C be sets. Show that

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

(c) $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$.

Solution. All three identities, and the two questions that follow, rest on a single observation, which we isolate once and then use without further comment.

Note that every element of every set appearing in this question is an ordered pair: each of the six sets is built from Cartesian products by union, intersection and difference, and each of these operations only removes elements. So in each of the three arguments below we may take an arbitrary element p of either side and write $p = (x, y)$ at the outset. Once that is done, Lemma P converts each membership statement into a statement about x and y, and the identity becomes a tautology of propositional logic.

Union. Let $p = (x, y)$. Then

$$\begin{aligned} p \in A \times (B \cup C) &\iff x \in A \text{ and } y \in B \cup C \\ &\iff x \in A \text{ and } (y \in B \text{ or } y \in C) \\ &\iff (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \\ &\iff p \in A \times B \text{ or } p \in A \times C \quad \iff \quad p \in (A \times B) \cup (A \times C), \end{aligned}$$

the third step being the distributivity of "and" over "or".

Intersection. Identical, with "or" replaced by "and" throughout; the third step is now the statement that "and" is associative and commutative, together with the fact that $x \in A$ may be repeated.

Difference. Let $p = (x, y)$. Then

$$\begin{aligned} p \in A \times (B \setminus C) &\iff x \in A \text{ and } (y \in B \text{ and } y \notin C) \\ &\iff (x \in A \text{ and } y \in B) \text{ and } \neg(x \in A \text{ and } y \in C) \\ &\iff p \in A \times B \text{ and } p \notin A \times C \quad \iff \quad p \in (A \times B) \setminus (A \times C). \end{aligned}$$

The second step deserves a word, since it is the only one that is not pure distributivity. In the forward direction we have $x \in A$ and $y \notin C$, and the latter alone refutes "$x \in A$ and $y \in C$". In the backward direction we have $x \in A$ and $y \in B$, and we must deduce $y \notin C$: if instead $y \in C$, then "$x \in A$ and $y \in C$" would hold, contrary to assumption.

Observe that the asymmetry of the difference identity is only apparent. What makes it work is that the left factor A is common to both products, so the hypothesis $x \in A$ is available on both sides of the negation.

Question (2).

Let A be a set, let I be a set, and for each $\alpha \in I$ let B_α be a set. Prove that

$$A \times \bigcup_{\alpha \in I} B_\alpha = \bigcup_{\alpha \in I} (A \times B_\alpha),$$

and that, provided $I \neq \emptyset$,

$$A \times \bigcap_{\alpha \in I} B_\alpha = \bigcap_{\alpha \in I} (A \times B_\alpha).$$

Show that the first identity holds also when $I = \emptyset$, and explain why the second cannot even be stated in that case.

Solution. First a remark on legitimacy. For each $\alpha \in I$ the object $A \times B_\alpha$ is a set, and it is determined by α; so $(A \times B_\alpha)_{\alpha\in I}$ is a family of sets indexed by I, obtained by applying replacement (Axiom 6) to I along $\alpha \mapsto A \times B_\alpha$. Hence the unions and intersections written above are legitimate objects, subject to the caveat about $\bigcap$ below.

The union. Every element of either side is an ordered pair: an element of the left side lies in a Cartesian product, and an element of the right side lies in $A \times B_\alpha$ for some α. So let $p = (x, y)$. Using Lemma P

$$\begin{aligned} p \in A \times \bigcup_{\alpha\in I} B_\alpha &\iff x \in A \ \text{ and } \ y \in \bigcup_{\alpha\in I} B_\alpha \\ &\iff x \in A \ \text{ and } \ \big(y \in B_\alpha \text{ for some } \alpha \in I\big) \\ &\iff \big(x \in A \text{ and } y \in B_\alpha\big) \text{ for some } \alpha \in I \\ &\iff p \in A \times B_\alpha \text{ for some } \alpha \in I \quad\iff\quad p \in \bigcup_{\alpha\in I}\big(A \times B_\alpha\big). \end{aligned}$$

The third step is the only one worth commenting on. Forwards: if $x \in A$ and $y \in B_\alpha$ for some particular α, that same α serves. Backwards: if for some α we have both $x \in A$ and $y \in B_\alpha$, then in particular $x \in A$, and that α witnesses $y \in \bigcup B_\beta$. The condition $x \in A$ does not mention α, which is what permits it to be moved across the quantifier.

The union when $I = \emptyset$. No object lies in $\bigcup_{\alpha\in\emptyset} B_\alpha$, so this union is $\emptyset$, and the left side is $A \times \emptyset = \emptyset$. By the same reasoning the right side, being a union over the empty index set, is $\emptyset$. So both sides are $\emptyset$ and the identity holds.

The intersection. Assume $I \neq \emptyset$ and fix some $\beta \in I$.

We first check that every element of either side is an ordered pair. For the left side this is Lemma P. For the right side, we have $\bigcap_{\alpha\in I}(A \times B_\alpha) \subseteq A \times B_\beta$, so every element of it is an ordered pair. *This is the first use of $I \neq \emptyset$.* Now let $p = (x, y)$. By Lemma P,

$$p \in A \times \bigcap_{\alpha\in I} B_\alpha \iff x \in A \ \text{ and } \ \big(y \in B_\alpha \text{ for all } \alpha \in I\big),$$

$$p \in \bigcap_{\alpha\in I}\big(A \times B_\alpha\big) \iff \big(x \in A \text{ and } y \in B_\alpha\big) \text{ for all } \alpha \in I.$$

It remains to see that the two right-hand conditions are equivalent.

($\Rightarrow$) Suppose $x \in A$ and $y \in B_\alpha$ for every α. Then for each α, both conjuncts hold. No hypothesis on I is needed.

($\Leftarrow$) Suppose that for every $\alpha \in I$ we have $x \in A$ and $y \in B_\alpha$. Instantiating at β gives $x \in A$; and the second conjunct, read for each α, gives $y \in B_\alpha$ for all α. *This instantiation is the second use of $I \neq \emptyset$:* without some β to instantiate at, the hypothesis would be vacuously true while the conclusion $x \in A$ would still have to be proved.

Why the statement fails to make sense when $I = \emptyset$. The failure is not that the identity is false; it is that neither side denotes anything. Recall: the intersection of a family is defined by first choosing some $\beta \in I$ and setting

$$\bigcap_{\alpha \in I} A_\alpha := \{\, x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I \,\},$$

which is a set by specification (Axiom 5). With $I = \emptyset$ there is no β to choose. *This is the third use of $I \neq \emptyset$.*

On the attempted induction. Suppose one proves, by induction on n, that $A \times (B_1 \cup \cdots \cup B_n) = (A \times B_1) \cup \cdots \cup (A \times B_n)$. This is a statement about a natural number n, and induction establishes it for every natural number. But $\bigcup_{\alpha \in I} B_\alpha$ for infinite I is not $B_1 \cup \cdots \cup B_n$ for any n: it is not built by iterating pairwise union at all, and there is no natural number for the induction to reach. The general identity must therefore be proved directly as above.

Question (3).

Let A, B, C be sets.

(a) Suppose $A \neq \emptyset$. Prove that $A \times B = A \times C$ implies $B = C$, and show that the hypothesis $A \neq \emptyset$ cannot be dropped.

(b) Prove that $A \times B = B \times A$ if and only if $A = B$ or $A = \emptyset$ or $B = \emptyset$.

Solution. *(a)* Suppose $A \neq \emptyset$ and $A \times B = A \times C$. Fix $a \in A$ as $A \neq \emptyset$.

Let $y \in B$. Then $(a, y) \in A \times B$ by Lemma P, hence $(a, y) \in A \times C$, hence $y \in C$ by Lemma P again. So $B \subseteq C$. Exchanging the roles of B and C gives $C \subseteq B$, so $B = C$.

The hypothesis cannot be dropped: take $A := \emptyset$, $B := \{0\}$ and $C := \{1\}$. Then $A \times B = \emptyset = A \times C$, but $B \neq C$, since $0 \in B$ and $0 \notin C$.

(b) ($\Leftarrow$) If $A = B$ the two products are the same set. If $A = \emptyset$ then both $A \times B$ and $B \times A$ are empty, hence equal; similarly if $B = \emptyset$.

($\Rightarrow$) Suppose $A \times B = B \times A$ and that A and B are both non-empty. Fix $a_0 \in A$ and $b_0 \in B$. Let $x \in A$. Then $(x, b_0) \in A \times B = B \times A$, so by Lemma P we get $x \in B$. Hence $A \subseteq B$. Let $y \in B$. Then $(a_0, y) \in A \times B = B \times A$, so by Lemma P we get $y \in A$. Hence $B \subseteq A$, and therefore $A = B$.

Question (4).

For sets B and C write $B \triangle C := (B \setminus C) \cup (C \setminus B)$ for the symmetric difference.

(a) Prove that $A \times (B \triangle C) = (A \times B) \triangle (A \times C)$ for all sets A, B, C.

(b) Show that $(A \times B) \triangle (C \times D) = (A \triangle C) \times (B \triangle D)$ is false in general.

Solution. *(a)* From Question (1), the operation $X \mapsto A \times X$ distributes over both $\cup$ and $\setminus$. Hence

$$\begin{aligned} A \times (B \triangle C) &= A \times \big((B \setminus C) \cup (C \setminus B)\big) \\ &= \big(A \times (B \setminus C)\big) \cup \big(A \times (C \setminus B)\big) && \text{(distribution over } \cup) \\ &= \big((A \times B) \setminus (A \times C)\big) \cup \big((A \times C) \setminus (A \times B)\big) && \text{(distribution over } \setminus \text{, twice)} \\ &= (A \times B) \triangle (A \times C). \end{aligned}$$

(b) Take $A := \{0\}$, $B := \{0\}$, $C := \{1\}$, $D := \{1\}$. Then $A \times B = \{(0,0)\}$ and $C \times D = \{(1,1)\}$. These are disjoint, so

$$(A \times B) \triangle (C \times D) = \{(0,0), (1,1)\},$$

a set of cardinality 2. On the other side, $A \triangle C = \{0,1\}$ and $B \triangle D = \{0,1\}$, so

$$(A \triangle C) \times (B \triangle D) = \{0,1\} \times \{0,1\},$$

which contains $(0,1)$. Since $(0,1) \neq (0,0)$ and $(0,1) \neq (1,1)$ by Eq. (3.5), the two sets are different.

An even more degenerate failure is available: take $A = C = \{0\}$, $B = \{0\}$, $D = \{1\}$. Then the left-hand side is $\{(0,0), (0,1)\}$, while $A \triangle C = \emptyset$ forces the right-hand side to be $\emptyset$.

Question (5).

Let A be a finite set with $|A| = a$, and let

$$T := \{ S \in \mathcal{P}(A) : |S| = 2 \}$$

be the collection of subsets of A having exactly two elements, a set by specification. Prove that T is finite and that, writing $t := |T|$,

$$2t + a = a^2.$$

Deduce that $t = 0$ if and only if $a \leq 1$.

Solution. We prove the following statement by induction on n:

> $Q(n)$: for every set A with $|A| = n$, the collection T of two-element subsets of A is finite and $2|T| + n = n^2$.

Base case. Let $|A| = 0$, so $A = \emptyset$. The only subset of $\emptyset$ is $\emptyset$ itself, which has cardinality 0, not 2. So $T = \emptyset$, which is finite with $|T| = 0$, and $2 \cdot 0 + 0 = 0 = 0^2$.

Inductive step. Assume $Q(n)$ and let A' be a set with $|A'| = n{+}{+}$. A' is non-empty; fix $x \in A'$ and put $A := A' \setminus \{x\}$, which has cardinality n. Let T' be the collection of two-element subsets of A', and split it as

$$U := \{ S \in T' : x \notin S \}, \qquad V := \{ S \in T' : x \in S \},$$

both sets by specification. Clearly $U \cap V = \emptyset$ and $U \cup V = T'$.

The piece U. We claim U is exactly the collection T of two-element subsets of A. If $S \in U$ then $S \subseteq A'$ and $x \notin S$, so $S \subseteq A' \setminus \{x\} = A$, and $|S| = 2$; hence $S \in T$. Conversely if $S \in T$ then $S \subseteq A \subseteq A'$ and $x \notin S$, since $x \notin A$; hence $S \in U$. By the induction hypothesis $U = T$ is finite and

$$2|U| + n = n^2.$$

The piece V. We claim the assignment $y \mapsto \{x, y\}$ is a bijection from A onto V.

It lands in V. Let $y \in A$. Then $y \neq x$, since $x \notin A$. Now $\{x, y\} = \{x\} \cup \{y\}$ is a union of two disjoint sets each of cardinality 1, so it is finite with cardinality $1 + 1 = 2$. Also $\{x, y\} \subseteq A'$, since $x \in A'$ and $y \in A \subseteq A'$. So $\{x, y\} \in V$.

Injective. Suppose $\{x,y\} = \{x,y'\}$ with $y, y' \in A$. Then $y \in \{x,y'\}$, so $y = x$ or $y = y'$. Since $y \neq x$, we get $y = y'$.

Surjective. Let $S \in V$, so $\lvert S \rvert = 2$ and $x \in S$. $S \setminus \{x\}$ has cardinality 1; in particular it is non-empty, so we may fix $y \in S \setminus \{x\}$, and then $\big(S \setminus \{x\}\big) \setminus \{y\}$ has cardinality 0 and is therefore empty. Hence $S \setminus \{x\} = \{y\}$.

Now $y \neq x$ and $y \in S \subseteq A'$, so $y \in A' \setminus \{x\} = A$. Finally $S = \{x,y\}$: both x and y lie in S; and if $z \in S$ with $z \neq x$ then $z \in S \setminus \{x\} = \{y\}$, so $z = y$.

Hence $\lvert V \rvert = \lvert A \rvert = n$.

Putting the pieces together. Since $T' = U \cup V$ with U and V disjoint and both finite, then T' is finite with

$$\lvert T' \rvert = \lvert U \rvert + n.$$

Therefore

$$2\lvert T' \rvert + n{+}{+} = 2\lvert U \rvert + 2n + n + 1 = \big(2\lvert U \rvert + n\big) + 2n + 1 = n^2 + 2n + 1 = (n+1)^2,$$

which is $Q(n{+}{+})$. This closes the induction.

Question (6).

For $y \in \mathbb{Q}$ write $\lfloor y \rfloor$ of y, and put $\{y\} := y - \lfloor y \rfloor$, the *fractional part* of y. Let $x \in \mathbb{Q}$ and let $N \in \mathbb{N}$ with $N \geq 1$.

(a) Prove that $0 \leq \{y\} < 1$ for every $y \in \mathbb{Q}$, and that for every rational t with $0 \leq t < 1$ there is exactly one $j \in \{0, 1, \ldots, N-1\}$ with

$$\frac{j}{N} \leq t < \frac{j+1}{N}.$$

(b) Prove that there are integers p and q with $1 \leq q \leq N$ and

$$\lvert qx - p \rvert < \frac{1}{N}.$$

(Apply the pigeonhole principle to the $N+1$ numbers $\{0 \cdot x\}, \{1 \cdot x\}, \ldots, \{Nx\}$ and the N intervals of part (a).)

Solution. *(a)* Note that, $\lfloor y \rfloor \leq y < \lfloor y \rfloor + 1$. Subtracting $\lfloor y \rfloor$ throughout, which preserves order, gives $0 \leq y - \lfloor y \rfloor < 1$, that is $0 \leq \{y\} < 1$. Now let $0 \leq t < 1$ and put $j := \lfloor Nt \rfloor$, so that $j \leq Nt < j+1$. Multiplying $0 \leq t < 1$ by the positive rational N gives $0 \leq Nt < N$.

$j \ge 0$. If $j \le -1$ then $j+1 \le 0 \le Nt$, contradicting $Nt < j+1$.

$j \le N-1$. From $j \le Nt$ and $Nt < N$ we get $j < N$, and for integers this gives $j \le N-1$.

So $j \in \{0, \dots, N-1\}$, and dividing $j \le Nt < j+1$ by the positive rational N gives $j/N \le t < (j+1)/N$.

Uniqueness: if j and j'' both work then $j/N \le t < (j+1)/N$ and $j''/N \le t < (j''+1)/N$. Multiplying by N gives $j \le Nt < j+1$ and $j'' \le Nt < j''+1$, and $j = j'' = \lfloor Nt \rfloor$.

(b) For $j = 0, 1, \dots, N-1$ put

$$A_j := \left\{ q \in \{0, 1, \dots, N\} \; : \; \tfrac{j}{N} \le \{qx\} < \tfrac{j+1}{N} \right\},$$

a set by specification. Each A_j is a subset of the finite set $\{0, \dots, N\}$ and so is finite.

Every q lies in exactly one A_j. By part (a), $0 \le \{qx\} < 1$, and again by part (a) there is exactly one $j \in \{0, \dots, N-1\}$ with $j/N \le \{qx\} < (j+1)/N$.

Hence $\bigcup_j A_j = \{0, 1, \dots, N\}$, which has $N+1$ elements, and there are N sets $A_0, \dots, A_{N-1}$. Since $N+1 > N$, the pigeonhole principle of Practice Problems 2, Question (7), gives some j with $\#(A_j) \ge 2$.

Choose two distinct elements of that A_j, say $q_1 < q_2$. Both $\{q_1 x\}$ and $\{q_2 x\}$ lie in the interval from j/N to $(j+1)/N$, so

$$\big|\{q_2 x\} - \{q_1 x\}\big| < \frac{j+1}{N} - \frac{j}{N} = \frac{1}{N}.$$

Now put

$$q := q_2 - q_1, \qquad p := \lfloor q_2 x \rfloor - \lfloor q_1 x \rfloor,$$

both integers. Then $1 \le q \le N$, since $0 \le q_1 < q_2 \le N$. And

$$\{q_2 x\} - \{q_1 x\} = \big(q_2 x - \lfloor q_2 x \rfloor\big) - \big(q_1 x - \lfloor q_1 x \rfloor\big) = (q_2 - q_1)x - \big(\lfloor q_2 x \rfloor - \lfloor q_1 x \rfloor\big) = qx - p,$$

so $|qx - p| < 1/N$, as required.

Question (7) [Exercise 5.3.3 of Tao].

Let a, b be rational numbers. Show that $a = b$ if and only if $\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b$ (that is, the constant Cauchy sequences $a, a, a, \dots$ and $b, b, b, \dots$ are equivalent if and only if $a = b$). This allows us to embed the rational numbers inside the real numbers in a well-defined manner.

Solution. Write $(a_n)_{n=1}^{\infty}$ for the constant sequence $a_n := a$, and similarly $b_n := b$.

Step 0: both formal limits are defined. A constant sequence is Cauchy: for any rational $\varepsilon > 0$ and any $j, k \geq 1$ we have $|a_j - a_k| = |a - a| = |0| = 0 \leq \varepsilon$. Since this holds for every $\varepsilon > 0$, $(a_n)_{n=1}^{\infty}$ is a Cauchy sequence. So $\mathrm{LIM}_{n\to\infty} a$ and $\mathrm{LIM}_{n\to\infty} b$ are real numbers.

Step 1: unwinding the definitions. The equation $\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b$ means that (a_n) and (b_n) are equivalent, which means: for every rational $\varepsilon > 0$ there is $N \geq 1$ with

$$|a_n - b_n| \leq \varepsilon \qquad \text{for all } n \geq N.$$

Here $a_n - b_n = a - b$ for every n, so the displayed condition neither depends on n nor on N: it holds for some N and all $n \geq N$ precisely when it holds outright. Hence

$$\mathrm{LIM}_{n\to\infty} a = \mathrm{LIM}_{n\to\infty} b \quad \iff \quad |a - b| \leq \varepsilon \ \text{ for every rational } \varepsilon > 0 \quad \iff \quad a = b.$$

Question (8) [Exercise 5.3.5 of Tao].

Show that $\mathrm{LIM}_{n\to\infty} 1/n = 0$.

Solution. *What is to be proved.* Let $a_n := 1/n$ for $n \geq 1$. This is a Cauchy sequence of rationals as proved in class, so $\mathrm{LIM}_{n\to\infty} a_n$ is a real number by definition of real numbers. The real number 0 is the formal limit $\mathrm{LIM}_{n\to\infty} b_n$ of the constant sequence $b_n := 0$, which is Cauchy since $|b_j - b_k| = 0 \leq \varepsilon$ for every $\varepsilon > 0$ and all j, k.

Note that the assertion $\mathrm{LIM}_{n\to\infty} a_n = \mathrm{LIM}_{n\to\infty} b_n$ means that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are equivalent. That is for every rational $\varepsilon > 0$ there exists $N \geq 1$ such that

$$|a_n - b_n| \leq \varepsilon \quad \text{for all } n \geq N.$$

Since $b_n = 0$ and $1/n$ is positive, $|a_n - b_n| = |1/n| = 1/n$. So we must produce, for each rational $\varepsilon > 0$, a natural number $N \geq 1$ with $1/n \leq \varepsilon$ for all $n \geq N$.

The proof. Let $\varepsilon > 0$ be rational. Then $1/\varepsilon$ is a rational number, so there is a natural number N with $1/\varepsilon < N < N + 1$. Clearly $N + 1 \geq 1$.

Let $n \geq N + 1$. Then

$$|a_n - b_n| = \frac{1}{n} \ \leq \ \frac{1}{N+1} \ < \ \varepsilon,$$

so in particular $|a_n - b_n| \leq \varepsilon$. As $\varepsilon > 0$ was an arbitrary rational, the two sequences are equivalent, and therefore

$$\mathrm{LIM}_{n\to\infty} \frac{1}{n} = 0.$$

Question (9).

For sequences (a_n) and (b_n) of rationals, write $(a_n) \approx (b_n)$ if for every $\varepsilon > 0$ there is $N \geq 1$ such that $|a_n - b_m| \leq \varepsilon$ *for all* $n, m \geq N$ — note that the two indices vary independently. Prove the following.

(a) $(a_n) \approx (a_n)$ if and only if (a_n) is Cauchy.

(b) If $(a_n) \approx (b_n)$ then both sequences are Cauchy, and (a_n) is equivalent to (b_n).

(c) Conversely, if (b_n) is Cauchy and (a_n) is equivalent to (b_n), then $(a_n) \approx (b_n)$.

(d) Exhibit equivalent sequences (a_n) and $v(b_n)$ such that $(a_n) \approx (b_n)$.

Solution. *(a)* Setting $b := a$ in the definition of $\approx$, the statement $(a_n) \approx (a_n)$ reads: for every $\varepsilon > 0$ there is $N \geq 1$ with $|a_n - a_m| \leq \varepsilon$ for all $n, m \geq N$. This is the definition of Cauchy sequence.

(b) Suppose $(a_n) \approx (b_n)$.

Both are Cauchy. Let $\varepsilon > 0$ and choose N for $\varepsilon/2$, so $|a_n - b_m| \leq \varepsilon/2$ for all $n, m \geq N$. For $j, k \geq N$, using the index $m := N$ twice,

$$|a_j - a_k| \leq |a_j - b_N| + |b_N - a_k| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

So (a_n) is Cauchy. Symmetrically, for $j, k \geq N$, $|b_j - b_k| \leq |b_j - a_N| + |a_N - b_k| \leq \varepsilon$, so (b_n) is Cauchy.

They are equivalent. Given $\varepsilon > 0$, take N as in the definition of $\approx$ and set $m := n$: for $n \geq N$ we get $|a_n - b_n| \leq \varepsilon$.

(c) Let $\varepsilon > 0$. Choose N_1 with $|a_n - b_n| \leq \varepsilon/2$ for $n \geq N_1$, which exists by equivalence; and N_2 with $|b_j - b_k| \leq \varepsilon/2$ for $j, k \geq N_2$, which exists because (b_n) is Cauchy. Put $N := \max(N_1, N_2)$. For $n, m \geq N$,

$$|a_n - b_m| \leq |a_n - b_n| + |b_n - b_m| \leq \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.$$

So $(a_n) \approx (b_n)$. Note that only (b_n) was required to be Cauchy.

Combining with (b): if both sequences are Cauchy then $\approx$ and equivalence of sequences hold or fail together.

(d) Take $a_n := b_n := (-1)^n$. The two sequences are equivalent, since $|a_n - b_n| = 0 \leq \varepsilon$ for every n and every $\varepsilon > 0$.

They do not satisfy $\approx$. Take $\varepsilon := 1$ and let $N \geq 1$ be arbitrary. Choose $n \geq N$ even and $m \geq N$ odd — for instance $n := 2N$ and $m := 2N + 1$. Then

$$|a_n - b_m| = |1 - (-1)| = 2 > 1 = \varepsilon.$$

So no N works, and $(a_n) \not\approx (b_n)$.

By part (a) this is exactly the statement that $\big((-1)^n\big)$ is not Cauchy, which is the reason the example works.

Question (10).

Call a sequence (a_n) *slowly varying* if for every $\varepsilon > 0$ there is $N \geq 1$ with

$$|a_{n+1} - a_n| \leq \varepsilon \qquad \text{for all } n \geq N.$$

(a) Prove that every Cauchy sequence is slowly varying.

(b) Let $a_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Prove that (a_n) is slowly varying but *not* Cauchy. *Hint: show $a_{2n} - a_n \geq \frac{1}{2}$ for every $n \geq 1$.*

Solution. *(a)* Let (a_n) be Cauchy and let $\varepsilon > 0$. Take N with $|a_j - a_k| \leq \varepsilon$ for all $j, k \geq N$. For $n \geq N$ we also have $n + 1 \geq N$, so $|a_{n+1} - a_n| \leq \varepsilon$.

(b) Slowly varying. Fix an $\varepsilon > 0$. Here $a_{n+1} - a_n = 1/(n+1)$, so $|a_{n+1} - a_n| = 1/(n+1) \leq 1/n$. Then there is an $N \geq 0$ with $1/n \leq \varepsilon$ for $n \geq N$.

The key inequality. Fix $n \geq 1$. Then

$$a_{2n} - a_n = \sum_{k=n+1}^{2n} \frac{1}{k}.$$

This sum has n terms, and each satisfies $1/k \geq 1/(2n)$, because $n+1 \leq k \leq 2n$ gives $0 < k \leq 2n$ and hence $1/k \geq 1/(2n)$. Summing this inequality over the n values of k gives

$$a_{2n} - a_n \;\geq\; n \cdot \frac{1}{2n} \;=\; \frac{1}{2}.$$

Not Cauchy. Suppose it were. Taking $\varepsilon := 1/4$ there would be $N \geq 1$ with $|a_j - a_k| \leq 1/4$ for all $j, k \geq N$. Take $j := 2N$ and $k := N$, both at least N. Then by the display above

$$|a_{2N} - a_N| = a_{2N} - a_N \geq \frac{1}{2} > \frac{1}{4},$$

a contradiction. So (a_n) is not Cauchy.

Question (11).

Call a sequence (a_n) *integrally Cauchy* if for every natural number $m \geq 1$ there is $N \geq 1$ with

$$|a_j - a_k| \leq \frac{1}{m} \qquad \text{for all } j, k \geq N.$$

Here ε has been restricted to run over the reciprocals of positive integers rather than over all positive rationals. Prove that a sequence is integrally Cauchy if and only if it is Cauchy.

Solution. *Cauchy* $\Rightarrow$ *integrally Cauchy.* Each $1/m$ with $m \geq 1$ is a positive rational, so the required N is supplied directly by the property that (a_j) is Cauchy.

Integrally Cauchy $\Rightarrow$ *Cauchy.* Let $\varepsilon > 0$ be a rational. For rational $1/\varepsilon$ there is an integer M with $M \leq 1/\varepsilon < M + 1$. Since $1/\varepsilon > 0$ we get $M + 1 \geq 1$, so $m := M + 1$ is a natural number with $m \geq 1$ and

$$m > \frac{1}{\varepsilon}.$$

Multiplying by the positive rational ε/m gives $\varepsilon > 1/m$.

Now apply the integrally Cauchy property at this m, obtaining $N \geq 1$ with $|a_j - a_k| \leq 1/m$ for all $j, k \geq N$. Then for such j, k,

$$|a_j - a_k| \leq \frac{1}{m} < \varepsilon,$$

so in particular $|a_j - a_k| \leq \varepsilon$. As ε was an arbitrary positive rational, (a_n) is Cauchy.