International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2026)

Practice Problems 1

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Disclaimer

Throughout, "Assignment 1, Question 1(d)" and the like refer to the parts of Question 1 of Assignment 1, where truncated subtraction $\ominus$ was introduced. As there, subtraction and division are not available; sums and products of natural numbers and their usual algebra may be used freely. Please note the following.

1. The solutions given here are not the only ones possible.
2. Use the solutions as guidelines.
3. If you find any mistakes or typos, you may reach out and it shall be cleared.
4. If something does not make sense in solutions, you may reach out and we shall clarify them in tutorials and/or classes.

Question (1). Let m and n be two natural numbers. Prove that if $n < m$ then $n^2 < m^2$, and deduce that if $n \leq m$ then $n^2 \leq m^2$.

Solution. Since $n < m$, write $m = n + k$ with $k \neq 0$. Then

$$m^2 = (n+k)(n+k) = n^2 + d, \qquad d := nk + nk + k^2.$$

The display alone gives $n^2 \leq m^2$, since m^2 is n^2 plus something. For strictness, suppose $n^2 = m^2$. Then $n^2 + 0 = n^2 + d$, so $d = 0$ by cancellation; a sum of natural numbers is zero only when every summand is, so in particular $k \times k = 0$ and hence $k = 0$, contradicting $k \neq 0$. Therefore $n^2 \neq m^2$, and combined with $n^2 \leq m^2$ this gives $n^2 < m^2$.

For the weak form, let $n \leq m$. Then either $n < m$, and the above gives $n^2 < m^2$, or $n = m$, and then $n^2 = m^2$. In both cases $n^2 \leq m^2$.

Question (2). Let $\mathbb{N}$ be the natural numbers. Prove that every $n \in \mathbb{N}$ satisfies $n \leq 1$ or $n \geq 2$.

Solution. We first Note two facts about the order, both of which are used again later in this sheet.
Lemma A. For all $n, m \in \mathbb{N}$: $n < m$ if and only if $n{+}{+} \leq m$.
Proof. If $n < m$, write $m = n + k$ with $k \neq 0$. Every nonzero natural is a successor, say $k = j{+}{+}$, so $m = n + j{+}{+} = (n + j){+}{+} = n{+}{+} + j$, whence $n{+}{+} \leq m$. Conversely, if $n{+}{+} \leq m$, write $m = n{+}{+} + j = n + j{+}{+}$; since $j{+}{+} \neq 0$ we get $n \leq m$ and $n \neq m$, that is $n < m$. $\square$
Lemma B. For all $n, m \in \mathbb{N}$: if $n{+}{+} \leq m{+}{+}$ then $n \leq m$.
Proof. Write $m{+}{+} = n{+}{+} + k = (n + k){+}{+}$. The successor is injective, so $m = n + k$, that is $n \leq m$. $\square$
Now let $n \in \mathbb{N}$. By trichotomy either $n \geq 2$, in which case we are done, or $n < 2$. In the latter case Lemma A gives $n{+}{+} \leq 2 = 1{+}{+}$, and Lemma B gives $n \leq 1$.

Question (3). Let $\mathbb{N}$ be the natural numbers. Prove that there is no $n \in \mathbb{N}$ with $n^2 = 2$.

Solution. Suppose such an n exists. By Question (2), either $n \leq 1$ or $n \geq 2$, and we treat the two cases.
Case $n \leq 1$. By Question (1), $n^2 \leq 1^2 = 1$. Since $1 < 2$, transitivity gives $n^2 < 2$, so $n^2 \neq 2$.

Case $n \geq 2$. By Question (1), $n^2 \geq 2^2 = 4$. Since $2 < 4$, we get $2 < n^2$, so again $n^2 \neq 2$.

Both cases contradict $n^2 = 2$.

Question (4). Prove that $2ab \leq a^2 + b^2$ for all $a, b \in \mathbb{N}$, with equality exactly when $a = b$.

Solution. The statement is symmetric in a and b, so by trichotomy we may assume $b \leq a$ and write $a = b + k$. Then

$$a^2 + b^2 = (b + k)^2 + b^2 = 2b^2 + 2bk + k^2 = 2b(b + k) + k^2 = 2ab + k^2,$$

so $a^2 + b^2 \geq 2ab$. Moreover $a^2 + b^2 = 2ab$ holds if and only if $k^2 = 0$, by cancellation, which holds if and only if $k = 0$, that is, if and only if $a = b$.

Question (5). Prove that $(n+1)^n \leq n^{n+1}$ for every $n \geq 3$, and show the hypothesis $n \geq 3$ cannot be weakened.

Solution. We use two order facts about $\mathbb{N}$: multiplication preserves order, so that $x \leq y$ implies $x^j \leq y^j$ for every j (induction on j); and multiplication by a nonzero factor may be cancelled across an inequality, that is, $xz \leq yz$ with $z \neq 0$ implies $x \leq y$.
Base. $4^3 = 64 \leq 81 = 3^4$.
Step. Let $n \geq 3$ and assume $(n+1)^n \leq n^{n+1}$. Since

$$(n+2)n = n^2 + 2n \leq n^2 + 2n + 1 = (n+1)^2,$$

raising to the power $n+1$ gives

$$(n+2)^{n+1}\, n^{n+1} \;=\; \big((n+2)n\big)^{n+1} \;\leq\; \big((n+1)^2\big)^{n+1} \;=\; (n+1)^{2n+2}.$$

Now split the exponent as $2n+2 = n + (n+2)$ and apply the induction hypothesis to the first factor:

$$(n+1)^{2n+2} = (n+1)^n\,(n+1)^{n+2} \;\leq\; n^{n+1}(n+1)^{n+2}.$$

Combining the two displays, $(n+2)^{n+1}n^{n+1} \leq n^{n+1}(n+1)^{n+2}$, and since $n \geq 3$ gives $n^{n+1} \neq 0$, cancellation yields

$$(n+2)^{n+1} \leq (n+1)^{n+2},$$

which is the assertion for $n+1$.
Sharpness. The hypothesis cannot be lowered even by one: for $n = 2$ the inequality fails, since $3^2 = 9 > 8 = 2^3$. It fails at the two smaller values as well, since $2^1 = 2 > 1 = 1^2$ and $1^0 = 1 > 0 = 0^1$. So $n \geq 3$ is not merely a convenient threshold; the statement is false at every smaller n.

Question (6). Prove that for all $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{N}$,

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \;\leq\; \Big(\sum_{i=1}^{n} a_i^2\Big)\Big(\sum_{i=1}^{n} b_i^2\Big).$$

Solution. We use that an inequality may be summed: if $x_t \geq y_t$ for every index t in a finite range, then $\sum_t x_t \geq \sum_t y_t$. This follows from the two-term case by induction on the number of terms, and is a fact of a different kind from the identities used below to rearrange sums.

Expanding both sides as double sums,

$$\Big(\sum_i a_i b_i\Big)^2 = \sum_i \sum_j a_i b_i a_j b_j, \qquad \Big(\sum_i a_i^2\Big)\Big(\sum_j b_j^2\Big) = \sum_i \sum_j a_i^2 b_j^2.$$

Each double sum splits into its diagonal part, over the indices with $i = j$, and its off-diagonal part, which we group into unordered pairs $\{i, j\}$ with $i \neq j$ by collecting the terms (i, j) and (j, i) together. This is a rearrangement of a finite sum, so it changes nothing.

On the diagonal the two sides agree term by term, both contributing $a_i^2 b_i^2$. For a pair $\{i, j\}$ with $i \neq j$, the left side contributes $2\, a_i b_i a_j b_j$ and the right side contributes $a_i^2 b_j^2 + a_j^2 b_i^2$. Put $u := a_i b_j$ and $v := a_j b_i$. Then $2uv = 2\, a_i b_i a_j b_j$ and $u^2 + v^2 = a_i^2 b_j^2 + a_j^2 b_i^2$, so Question (4) gives exactly

$$2\, a_i b_i a_j b_j \;=\; 2uv \;\leq\; u^2 + v^2 \;=\; a_i^2 b_j^2 + a_j^2 b_i^2.$$

Summing these inequalities over all unordered pairs, and adding the equal diagonal terms, gives the result.

Question (7). Define $! : \mathbb{N} \to \mathbb{N}$ by recursion:

$$0! := 1, \qquad (n{+}{+})! := n! \times n{+}{+}.$$

Prove that $(n!)^2 \geq n^n$ for every $n \in \mathbb{N}$.

Solution. Products with no factors are taken to be 1, and $x^0 := 1$ by the recursive definition of exponentiation, so in particular $0^0 = 1$. No subtraction is used anywhere below: the partner of an index i is characterised by an equation, not by naming $n - i$.

Lemma A. $n! = \prod_{i=1}^{n} i$ for every $n \in \mathbb{N}$.

Proof. Induction on n. For $n = 0$ both sides are 1, the left by the first clause of the definition and the right because the product is empty. Assuming the identity for n,

$$(n{+}{+})! = n! \times n{+}{+} = \Big(\prod_{i=1}^{n} i\Big) \times n{+}{+} = \prod_{i=1}^{n{+}{+}} i. \qquad \square$$

Lemma B (partners). Fix $n \geq 1$ and let $1 \leq i \leq n$. Then there is exactly one $j \in \mathbb{N}$ with $i + j = n{+}{+}$, and it satisfies $1 \leq j \leq n$. Writing $\sigma(i)$ for this j, the map

σ sends $\{1, \dots, n\}$ to itself and satisfies $\sigma(\sigma(i)) = i$; in particular σ is a bijection of $\{1, \dots, n\}$.

Proof. Since $i \leq n$ and $n < n{+}{+}$, we have $i \leq n{+}{+}$, so $n{+}{+} = i + j$ for some j, and j is unique by cancellation. If $j = 0$ then $i = n{+}{+}$, contradicting $i \leq n < n{+}{+}$; so $j \neq 0$, that is $1 \leq j$. Also $j \leq i + j = n{+}{+}$ and $j \neq n{+}{+}$, since $i \neq 0$; so $j < n{+}{+}$, and Lemmas A and B of Question (2) give $j \leq n$. The identity $\sigma(i) + i = n{+}{+}$ is the same equation read with the summands exchanged, so $\sigma(\sigma(i)) = i$. A map that is its own inverse is a bijection. $\square$

Lemma C. If $i + j = n{+}{+}$ with $i \neq 0$ and $j \neq 0$, then $i \times j \geq n$.

Proof. Every nonzero natural is a successor, so write $i = s{+}{+}$ and $j = t{+}{+}$. Then $s{+}{+} + t{+}{+} = (s+t){+}{+}{+}{+} = n{+}{+}$, and injectivity of the successor gives $(s+t){+}{+} = n$. Hence

$$i \times j = s{+}{+} \times t{+}{+} = s \times t + s + t + 1 = s \times t + (s + t){+}{+} = s \times t + n \;\geq\; n. \qquad \square$$

Proof of the claim. For $n = 0$ both sides equal 1. Let $n \geq 1$. By Lemma A,

$$(n!)^2 = \Big(\prod_{i=1}^{n} i\Big)\Big(\prod_{j=1}^{n} j\Big).$$

By Lemma B, σ is a bijection of $\{1, \dots, n\}$, so reindexing the second product along σ leaves it unchanged: $\prod_{j=1}^{n} j = \prod_{i=1}^{n} \sigma(i)$. Therefore

$$(n!)^2 = \prod_{i=1}^{n} \big(i \times \sigma(i)\big).$$

Each factor satisfies $i + \sigma(i) = n{+}{+}$ with both i and $\sigma(i)$ nonzero, so Lemma C gives $i \times \sigma(i) \geq n$. Since a product of natural numbers preserves $\geq$ factor by factor,

$$(n!)^2 = \prod_{i=1}^{n} \big(i \times \sigma(i)\big) \;\geq\; \prod_{i=1}^{n} n = n^n. \qquad \square$$

Remark: an alternative proof straight from the recursion. One can avoid products altogether and induct, using Question (5). For $n = 0$ the claim reads $1 \geq 1$. Assume $(n!)^2 \geq n^n$. Then

$$\big((n{+}{+})!\big)^2 = (n!)^2 (n{+}{+})^2 \;\geq\; n^n (n{+}{+})^2,$$

so it suffices to show $n^n (n{+}{+})^2 \geq (n{+}{+})^{n{+}{+}} = (n{+}{+})^n (n{+}{+})$, that is, $n^n (n{+}{+}) \geq (n{+}{+})^n$. For $n \geq 3$ this follows from Question (5), since $(n{+}{+})^n \leq n^{n{+}{+}} = n^n \times n \leq$

$n^n \times n{+}{+}$. The three remaining cases are immediate: $1 \geq 1$ for $n = 0$, $2 \geq 2$ for $n = 1$, and $12 \geq 9$ for $n = 2$.

Question (8). *(Backwards induction.)* Let $P(m)$ be a property pertaining to the natural numbers with the following feature:

$$\text{whenever } P(m{+}{+}) \text{ is true, } P(m) \text{ is true.} \tag{$*$}$$

Let $n \in \mathbb{N}$ and suppose $P(n)$ is true. Prove that $P(m)$ is true for every natural number $m \leq n$.

Solution. Fix a property P satisfying $(*)$. For $n \in \mathbb{N}$ let $Q(n)$ be the statement

$$Q(n): \quad \text{if } P(n) \text{ is true, then } P(m) \text{ is true for every } m \leq n.$$

We prove $Q(n)$ for all n by induction on n. Note that the induction is on n and the statement being inducted is Q, not P.

Base case. We show $Q(0)$. Suppose $P(0)$ is true and let $m \leq 0$, say $0 = m + a$. Then $m = 0$, since a sum of natural numbers is zero only when both summands are. So $P(m)$ is $P(0)$, which is true.

Inductive step. Suppose $Q(n)$ holds; we show $Q(n{+}{+})$. So assume $P(n{+}{+})$ is true, and let $m \leq n{+}{+}$ be arbitrary.
Applying $(*)$ with $m := n$, the truth of $P(n{+}{+})$ gives the truth of $P(n)$. Since $Q(n)$ holds, $P(k)$ is true for every $k \leq n$.
Now $m \leq n{+}{+}$ means $m < n{+}{+}$ or $m = n{+}{+}$.
If $m = n{+}{+}$, then $P(m)$ is $P(n{+}{+})$, true by assumption.
If $m < n{+}{+}$, then $m{+}{+} \leq n{+}{+}$ by Lemma A of Question (2), and $m \leq n$ by Lemma B of Question (2); hence $P(m)$ is true by the previous paragraph.
In both cases $P(m)$ is true, so $Q(n{+}{+})$ holds.

By the principle of induction, $Q(n)$ is true for every $n \in \mathbb{N}$. Applying this to the given n, for which $P(n)$ is assumed true, yields $P(m)$ for all $m \leq n$.

Question (9). Let $\mathcal{A}$ be a set of sets. Prove that $\bigcap \mathcal{A}$ is a set provided $\mathcal{A} \neq \emptyset$, and explain what fails when $\mathcal{A} = \emptyset$.

Solution. Suppose $\mathcal{A} \neq \emptyset$ and fix some $A_0 \in \mathcal{A}$. By specification (axiom 5),

$$\bigcap \mathcal{A} := \{\, x \in A_0 : x \in S \text{ for all } S \in \mathcal{A} \,\}$$

is a set. It has the right elements: if $x \in S$ for every $S \in \mathcal{A}$ then in particular $x \in A_0$, so no elements are lost. It is also independent of the choice of A_0, since the defining condition already forces membership in every element of $\mathcal{A}$.

If $\mathcal{A} = \emptyset$ there is no A_0 to specify from, and this is not a defect of the proof. The condition "$x \in S$ for all $S \in \emptyset$" is vacuously true of *every* object, so $\bigcap \emptyset$ would have to be the set of all sets. No such set exists, by Russell's paradox as shown in class, so no set has the elements $\bigcap \emptyset$ would need.

Question (10). Exhibit a set A and a property $P(x, y)$ for which the conclusion of replacement (axiom 6) fails once the requirement "at most one y" is dropped.

Solution. Take $A := \{\emptyset\}$ and

$$P(x, y) : \quad y = y,$$

which every object satisfies. For $x = \emptyset$ there are many y with $P(x, y)$, so the hypothesis fails. The purported conclusion would be that

$$\{\, y : P(x, y) \text{ for some } x \in A \,\} = \{\, y : y \text{ is a set} \,\}$$

is a set — that is, a set of all sets. By Russell's paradox as shown in class, no such set exists, so the conclusion of replacement is false without the single-valuedness requirement.

Question (11). Prove that there is no set X with $\mathcal{P}(X) \subseteq X$.

Solution. Suppose $\mathcal{P}(X) \subseteq X$. Let $R := \{x \in X : x \notin x\}$, a set by specification (axiom 5). Then $R \subseteq X$, so $R \in \mathcal{P}(X) \subseteq X$, giving $R \in X$. Hence, using $R \in X$ for the second equivalence,

$$R \in R \iff (R \in X \text{ and } R \notin R) \iff R \notin R,$$

a contradiction. This is Russell's construction carried out inside X.

Question (12). Prove that $\mathcal{S} := \{\, \{x\} : x \text{ is a set} \,\}$ is not a set.

Solution. Suppose $\mathcal{S}$ were a set. Every element of $\mathcal{S}$ is a set, so the union axiom applies and $\bigcup \mathcal{S}$ is a set. For any set x,

$$x \in \bigcup \mathcal{S} \iff x \in T \text{ for some } T \in \mathcal{S} \iff x \in \{y\} \text{ for some set } y,$$

and taking $y := x$ shows this holds for every set x. So $\bigcup \mathcal{S}$ would be a set of all sets, and no such set exists by Russell's paradox as shown in class. Hence $\mathcal{S}$ is not a set.

Question (13). Let $f : X \to Y$ be bijective. For every $y \in Y$ there is exactly one $x \in X$ with $f(x) = y$. Denoting this x by $f^{-1}(y)$ defines a function $f^{-1} : Y \to X$, called the *inverse* of f. By construction,

$$f^{-1}(y) = x \quad \text{if and only if} \quad f(x) = y. \tag{$\dagger$}$$

Recall also that $f : X \to Y$ is *invertible* if there is a function $g : Y \to X$ with $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$; such a g is called an inverse of f. Answer the following questions.

(a) Let $f : X \to Y$ be bijective with inverse f^{-1}. Then show that $f^{-1}(f(x)) = x$ for all $x \in X$, and $f(f^{-1}(y)) = y$ for all $y \in Y$.

(b) Show that f^{-1} is bijective.

(c) Conclude that f^{-1} is invertible with inverse f; that is, $(f^{-1})^{-1} = f$.

(d) Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are bijective, then so is $g \circ f$, and that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Solution. (a) Let $x \in X$ and put $y := f(x)$. Then $f(x) = y$, so by ($\dagger$) we have $f^{-1}(y) = x$, that is, $f^{-1}(f(x)) = x$.
Let $y \in Y$ and put $x := f^{-1}(y)$. Then $f^{-1}(y) = x$, so by ($\dagger$) read in the other direction, $f(x) = y$, that is, $f(f^{-1}(y)) = y$.

(b) *Injective.* Suppose $f^{-1}(y) = f^{-1}(y')$ for $y, y' \in Y$. Applying f to both sides and using the second law in (a),

$$y = f\big(f^{-1}(y)\big) = f\big(f^{-1}(y')\big) = y'.$$

Surjective. Let $x \in X$. Then $f(x) \in Y$, and by the first law in (a), $f^{-1}(f(x)) = x$. So x is a value of f^{-1}.

(c) We first note that inverses, in the sense of invertibility, are unique.
Lemma. Let $F : Y \to X$ and suppose $g, h : X \to Y$ are both inverses of F. Then $g = h$.

Proof. Both g and h are functions from X to Y, so it suffices to compare values. Let $x \in X$ and put $u := g(x) \in Y$. Since g is an inverse of F we have $F(u) = F(g(x)) = x$, and since h is an inverse of F we have $h(F(u)) = u$. Therefore

$$h(x) = h\big(F(u)\big) = u = g(x). \qquad \square$$

Now the two laws of part (a) say precisely that f is an inverse of f^{-1} in the sense of invertibility: $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$, read with f^{-1} in the role of the given function. By part (b), f^{-1} is bijective, so it has an inverse function $(f^{-1})^{-1} : X \to Y$, and applying part (a) to f^{-1} shows that $(f^{-1})^{-1}$ is also an inverse of f^{-1}. By the Lemma, applied with $F := f^{-1}$, these coincide: $(f^{-1})^{-1} = f$.

(d) $g \circ f$ *is injective.* Suppose $(g \circ f)(x) = (g \circ f)(x')$, that is, $g(f(x)) = g(f(x'))$. Since g is injective, $f(x) = f(x')$; since f is injective, $x = x'$.
$g \circ f$ *is surjective.* Let $z \in Z$. Since g is surjective there is $y \in Y$ with $g(y) = z$, and since f is surjective there is $x \in X$ with $f(x) = y$. Then $(g \circ f)(x) = g(f(x)) = g(y) = z$.
So $g \circ f$ is bijective and $(g \circ f)^{-1} : Z \to X$ exists.

The formula. By the Lemma of part (c), it suffices to check that $f^{-1} \circ g^{-1}$ satisfies the two cancellation laws for $g \circ f$, since $(g \circ f)^{-1}$ satisfies them by part (a) and inverses are unique.
For $x \in X$, using the laws for g and then for f,

$$\big(f^{-1} \circ g^{-1}\big)\big((g \circ f)(x)\big) = f^{-1}\Big(g^{-1}\big(g(f(x))\big)\Big) = f^{-1}\big(f(x)\big) = x.$$

For $z \in Z$, using the laws for f and then for g,

$$(g \circ f)\Big(\big(f^{-1} \circ g^{-1}\big)(z)\Big) = g\Big(f\big(f^{-1}(g^{-1}(z))\big)\Big) = g\big(g^{-1}(z)\big) = z.$$

Hence $f^{-1} \circ g^{-1}$ is an inverse of $g \circ f$, and by uniqueness it is *the* inverse.

Question (14). For $v \in \mathbb{N}$ define the *translation by* v

$$\tau_v : \mathbb{N} \to \mathbb{N}, \qquad \tau_v(n) := n + v.$$

For $E \subseteq \mathbb{N}$, the *image* and the *preimage* of E under τ_v are

$$\tau_v(E) := \{\, n + v : n \in E \,\}, \qquad \tau_v^{-1}(E) := \{\, n \in \mathbb{N} : n + v \in E \,\}.$$

The preimage is defined for every function; no inverse function is presumed to exist. Finally, for $a \le b$ write

$$I_{a,b} := \{\, i \in \mathbb{N} : a \le i \le b \,\},$$

and call $b \ominus a$ the *length* of $I_{a,b}$, where $\ominus$ is the truncated subtraction of Question 1 of Assignment 1. Note that length is defined only for sets of the form $I_{a,b}$ with $a \le b$, and in particular not for the empty set.
Prove the following.

(a) $\tau_v \circ \tau_w = \tau_{v+w}$ for all $v, w \in \mathbb{N}$, and τ_0 is the identity map on $\mathbb{N}$. Show that τ_v is injective, and that it is surjective if and only if $v = 0$.

(b) For every $E \subseteq \mathbb{N}$,

$$\tau_v^{-1}(E) = \{\, n \ominus v : n \in E \text{ and } v \le n \,\}.$$

(c) Let $a \le b$. Show that

$$\tau_v^{-1}\big(I_{a,b}\big) = \begin{cases} I_{a\ominus v,\ b\ominus v} & \text{if } v \le b, \\ \emptyset & \text{if } b < v. \end{cases}$$

(d) Show that $\tau_v(I_{a,b}) = I_{a+v,\,b+v}$, and that this always has the same length as $I_{a,b}$. Deduce from (c) that $\tau_v^{-1}(I_{a,b})$ is an interval of the same length as $I_{a,b}$ if and only if $v \le a$.

Solution. References to parts (b)–(f) below are to Question 1 of Assignment 1. We shall use two further facts about $\ominus$, which we recall first.

Lemma M (monotonicity). If $a \le b$ then $a \ominus v \le b \ominus v$.
Proof. If $a < v$ then $a \le v$, so $a \ominus v = 0$ by Assignment 1, Question 1(e), and $0 \le b \ominus v$. If $v \le a$ then also $v \le b$, and Assignment 1, Question 1(f) lets us write $a = (a \ominus v) + v$ and $b = (b \ominus v) + v$. From $a \le b$ we get $(a \ominus v) + v \le (b \ominus v) + v$, and cancelling v gives $a \ominus v \le b \ominus v$. $\square$

Lemma S (strict decrease). If $a < v \le b$ then $b \ominus v < b \ominus a$.
Proof. Write $v = a + u$ with $u \ne 0$, and $b = v + w$. Then $b = w + v$, so $b \ominus v = w$ by Assignment 1, Question 1(d); and $b = (u + w) + a$, so $b \ominus a = u + w$ by the same part. Since $u \ne 0$ we have $w \ne u + w$, and $w \le u + w$, so $w < u + w$. $\square$

Lemma T (translation invariance of length). For all $x, y, v \in \mathbb{N}$, $(x + v) \ominus (y + v) = x \ominus y$.

Proof. By Assignment 1, Question 1(c) and then 1(d),

$$(x+v) \ominus (y+v) = (x+v) \ominus (v+y) = \big((x+v) \ominus v\big) \ominus y = x \ominus y. \qquad \square$$

(a) For every n, $(\tau_v \circ \tau_w)(n) = (n+w)+v = n+(v+w) = \tau_{v+w}(n)$, and $\tau_0(n) = n+0 = n$. If $\tau_v(n) = \tau_v(m)$ then $n+v = m+v$, so $n = m$ by cancellation; hence τ_v is injective. If $v = 0$ then τ_0 is the identity, which is surjective. If $v \neq 0$ then $n+v \geq v > 0$ for every n, so 0 is not a value of τ_v and τ_v is not surjective.

(b) ($\subseteq$) Let $m \in \tau_v^{-1}(E)$ and put $n := m+v$. Then $n \in E$ and $v \leq n$, and $n \ominus v = (m+v) \ominus v = m$ by Assignment 1, Question 1(d). So m belongs to the right-hand set.
($\supseteq$) Let $n \in E$ with $v \leq n$, and put $m := n \ominus v$. By Assignment 1, Question 1(f), $m+v = (n \ominus v)+v = n$, which lies in E; hence $m \in \tau_v^{-1}(E)$.

(c) Let $m \in \mathbb{N}$. By definition, $m \in \tau_v^{-1}(I_{a,b})$ holds exactly when $a \leq m+v$ and $m+v \leq b$. We treat the two conditions separately.
The lower condition. We claim $a \leq m+v$ if and only if $a \ominus v \leq m$. If $v \geq a$ then $a \ominus v = 0$ by Assignment 1, Question 1(e), and also $m+v \geq v \geq a$, so both sides hold for every m. If $v < a$, then $v \leq a$, so by Assignment 1, Question 1(f) we may write $a = (a \ominus v)+v$; hence $a \leq m+v$ is equivalent to $(a \ominus v)+v \leq m+v$, which by cancellation is equivalent to $a \ominus v \leq m$.
The upper condition. If $b < v$ then $m+v \geq v > b$ for every m, so no m qualifies and $\tau_v^{-1}(I_{a,b}) = \emptyset$. If $v \leq b$ then $b = (b \ominus v)+v$ by Assignment 1, Question 1(f), so $m+v \leq b$ is equivalent to $m+v \leq (b \ominus v)+v$, that is, to $m \leq b \ominus v$.
Combining, when $v \leq b$ the two conditions read $a \ominus v \leq m \leq b \ominus v$, which is exactly membership in $I_{a \ominus v,\, b \ominus v}$. This interval is legitimate: $a \leq b$ gives $a \ominus v \leq b \ominus v$ by Lemma M.

(d) *The image.* We check the two inclusions directly; this does not follow from (c). If $i \in I_{a,b}$ then $a \leq i \leq b$, so $a+v \leq i+v \leq b+v$ and $\tau_v(i) \in I_{a+v,\,b+v}$. Conversely, let $m \in I_{a+v,\,b+v}$. Then $v \leq a+v \leq m$, so Assignment 1, Question 1(f) gives $m = i+v$ with $i := m \ominus v$. From $a+v \leq i+v$ and $i+v \leq b+v$, cancellation gives $a \leq i \leq b$, so $i \in I_{a,b}$ and $m = \tau_v(i)$. Hence $\tau_v(I_{a,b}) = I_{a+v,\,b+v}$, which is a legitimate interval since $a+v \leq b+v$. Its length is

$$(b+v) \ominus (a+v) = b \ominus a$$

by Lemma T, the same as the length of $I_{a,b}$.
The preimage. There are three cases.

If $v \leq a$, then $v \leq b$ as well, so by (c) the preimage is $I_{a \ominus v,\, b \ominus v}$. Writing $a = (a \ominus v) + v$ and $b = (b \ominus v) + v$ by Assignment 1, Question 1(f), Lemma T gives

$$b \ominus a = \big((b \ominus v) + v\big) \ominus \big((a \ominus v) + v\big) = (b \ominus v) \ominus (a \ominus v),$$

so the preimage has the same length as $I_{a,b}$.
If $a < v \leq b$, then $a \ominus v = 0$ by Assignment 1, Question 1(e), so by (c) the preimage is $I_{0,\, b \ominus v}$, whose length is $(b \ominus v) \ominus 0 = b \ominus v$. By Lemma S, $b \ominus v < b \ominus a$, so the length has strictly decreased.
If $b < v$, the preimage is empty by (c), and the empty set is not an interval $I_{c,d}$ with $c \leq d$, since every such interval contains c. So it has no length, and in particular is not an interval of the same length as $I_{a,b}$.
Hence $\tau_v^{-1}(I_{a,b})$ is an interval of the same length as $I_{a,b}$ precisely when $v \leq a$.