International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2026)

Assignment 1

Due date: **September 11, 2026** Total Marks: 75

Disclaimer

The level of depth and difficulty in this assignment is **not indicative of the upcoming quiz 1**. Its purpose is to provide practice with key concepts covered and push towards better understanding of notions. Here are some guidelines for the solutions.

1. The algebra of natural numbers is allowed but one cannot use subtraction, division, etc.
2. The axioms used to solve any question must be stated at least once in the solution.
3. The use of any axiom must be demonstrated clearly. Failing this will result in deduction of marks. For example, saying that "Axiom 5 yields this result" is a meaningless statement unless it is shown as to how is axiom 5 invoked.

Question 1 [15 Marks]. Define $\mathsf{pred} : \mathbb{N} \to \mathbb{N}$ recursively by

$$\mathsf{pred}(0) := 0, \qquad \mathsf{pred}(n{+}{+}) := n,$$

and then define $m \ominus n$ ("m monus n") by recursion on n:

$$m \ominus 0 := m, \qquad m \ominus (n{+}{+}) := \mathsf{pred}(m \ominus n).$$

Prove the following.

(a) [**1 Mark**] For all $n \in \mathbb{N}$, $0 \ominus n = 0$.

(b) [**1 Mark**] For all $m, n \in \mathbb{N}$, $m{+}{+} \ominus n{+}{+} = m \ominus n$.

(c) **[2 Marks]** $(m \ominus n) \ominus k = m \ominus (n + k)$ for all $m, n, k \in \mathbb{N}$.

(d) **[1 Mark]** $(m + n) \ominus n = m$ for all $m, n \in \mathbb{N}$.

(e) **[2 Marks]** $m \ominus n = 0$ if and only if $m \leq n$.

(f) **[2 Marks]** $(m \ominus n) + n = m$ if and only if $n \leq m$.

(g) **[2 Marks]** Define $\max(m, n) := m$ if $n \leq m$, and $\max(m, n) := n$ otherwise. Then $\max(m, n) = m + (n \ominus m)$, and deduce that $m + (n \ominus m) = n + (m \ominus n)$.

(h) **[2 Marks]** $k \times (m \ominus n) = (k \times m) \ominus (k \times n)$.

(i) **[2 Marks]** Suppose $k \leq m$ and $k \leq n$. Prove that $m \ominus k = n \ominus k$ implies $m = n$. Show that neither hypothesis can be dropped.

Rationale.

Monus is subtraction that stops at 0, so that it never leaves $\mathbb{N}$. Parts (a)–(d) are pure induction; from (e) onwards the technique changes to trichotomy together with writing the larger number as the smaller plus something, and each of (e)–(i) leans on the parts before it rather than returning to the definition. The point of the question is the contrast between (d) and (f): adding and then truncating always returns the original number, whereas truncating and then adding does so only when nothing was lost. Part (g) is the payoff — max, an order-theoretic notion, turns out to be definable from $+$ and $\ominus$ alone.

A check. Note that the clauses defining pred read like a definition by cases rather than a recursion, since the right-hand side never mentions $\mathsf{pred}(n)$; splitting on whether a number is 0 or a successor is not itself an axiom, so it is worth checking that these clauses do fall under the recursion theorem. Question 2 is where the corresponding check fails.

Question 2 [10 Marks]. We wish to define $h : \mathbb{N} \to \mathbb{N}$, the "lower half" of n, by

$$h(0) := 0, \qquad h(0{+}{+}) := 0, \qquad h(n{+}{+}{+}{+}) := h(n){+}{+}. \tag{1}$$

(a) **[2 Marks]** Prove that every $n \in \mathbb{N}$ is of exactly one of the forms 0, $0{+}{+}$, or $m{+}{+}{+}{+}$, and that in the last case m is uniquely determined by n. Explain why this is needed before Eq. (1) can be read as a definition at all.

(b) **[2 Marks]** Explain why Eq. (1) is nevertheless not an instance of the recursion theorem.

(c) [**4 Marks**] Prove that a function $h : \mathbb{N} \to \mathbb{N}$ satisfying Eq. (1) exists. State clearly which set you recurse into, what the step rule is, and how h itself is recovered from the function the recursion theorem gives you.

(d) [**2 Marks**] Prove that such an h is unique.

Rationale.

Two different things can go wrong with a proposed definition, and this question separates them. The first is whether the clauses assign a value to every natural number, and never two: that is part (a), and it is the same obligation discharged for pred in Question 1. The second is whether the clauses fall under the recursion theorem, which determines the value at $n{+}{+}$ from the value at n alone, whereas Eq. (1) reaches two steps back. The repair is to recurse on *pairs*, carrying one step of history along, and this uses the fact that $\mathbb{N} \times \mathbb{N}$ is a set; the function so obtained is not h, and part of the work in (c) is getting h back out of it.

Question 3 [8 Marks]. Let $n = 2^k$ for some $k \in \mathbb{N}$. Prove that for all $a_1, \ldots, a_n \in \mathbb{N}$,

$$(a_1 + a_2 + \cdots + a_n)^n \ \geq \ n^n \, a_1 a_2 \cdots a_n.$$

Rationale.

This is the arithmetic–geometric mean inequality with the denominators cleared, so that it becomes a statement about natural numbers. The induction is on k rather than on n: each step doubles the number of variables. Restricting to powers of two is what keeps the induction purely forward; the classical route to arbitrary n descends from 2^k by replacing a variable with the average of the others, and averages do not exist in $\mathbb{N}$.

Question 4 [8 Marks]. Suppose $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ are natural numbers. Prove that

$$n \sum_{i=1}^{n} a_i b_i \ \geq \ \Big(\sum_{i=1}^{n} a_i\Big)\Big(\sum_{i=1}^{n} b_i\Big).$$

Rationale.

No induction on n is needed here, in contrast with Question 3; finite sums and products of natural numbers, together with their usual algebra, may be taken as given. Everything follows from a single pairwise inequality, proved by writing each larger term as the smaller plus something, and then summing that inequality over all pairs of indices — and summing an *inequality* over many terms is a fact that should be stated rather than assumed silently, since it is of a different kind from the identities used to rearrange the sums.

Question 5 [10 Marks]. Show that there is no function $f : \mathbb{N} \to \mathbb{N}$ satisfying

$$f(f(n)) = n{+}{+} \qquad \text{for every } n \in \mathbb{N},$$

but that there is one satisfying $f(f(n)) = n{+}{+}{+}{+}$ for every n.

Rationale.

A functional equation, rather than an identity to be verified. The work lies in showing that any such f must satisfy $f(n{+}{+}) = f(n){+}{+}$ and hence be a translation; after that the obstruction is arithmetic, since no natural number added to itself gives 1. The second half is deliberately easy: a "square root" of the double successor exists at once, so the failure in the first half is genuinely about the number 1 and not about composition.

Question 6 [8 Marks]. Let X be a set. Prove that $\{X, \mathcal{P}(X), \mathcal{P}(\mathcal{P}(X)), \ldots\}$ is a set.

Rationale.

The ellipsis in the statement is not an abbreviation that may be taken for granted; supplying its meaning is the first half of the question, and a solution that begins by treating the collection as given has assumed what it was asked to produce. So the first task is to say precisely which objects the ellipsis stands for, and only then to ask whether they form a set. Specification (axiom 5) is unavailable for the second task, since there is no set known in advance to contain all the iterated power sets — producing one is exactly the problem. Replacement (axiom 6) is what transports $\mathbb{N}$ along the rule that the ellipsis names. A second subtlety is worth noticing, and it is the reason the first task is not merely bookkeeping: defining that rule is itself a recursion with no target set given in advance, so the recursion theorem as stated does not apply to it unaltered. Replacement is therefore doing double duty. This is the standard example of a set whose existence genuinely requires replacement.

Question 7 [6 Marks]. Let X be a set. Prove that no function $f : X \to \mathcal{P}(X)$ is surjective.

Rationale.

This is a deep result: it produces infinitely many different sizes of infinity. In it, a set needs to be built to disagree with $f(x)$ at x for every x at once, so it can be no value of f. It is worth comparing the argument with Russell's paradox — they are the same construction in different clothing, and recognising this is part of the exercise.

Question 8 [10 Marks]. If X is a subset of Y, let $\mathsf{inc}_{X\to Y} : X \to Y$ be the *inclusion map* from X to Y, defined by

$$\mathsf{inc}_{X\to Y}(x) := x \qquad \text{for all } x \in X.$$

The map $\mathsf{inc}_{X\to X}$ is called the *identity map* on X. Prove the following.

(a) **[2 Marks]** If $X \subseteq Y \subseteq Z$, then $\mathsf{inc}_{Y\to Z} \circ \mathsf{inc}_{X\to Y} = \mathsf{inc}_{X\to Z}$.

(b) **[2 Marks]** If $f : A \to B$ is any function, then $f = f \circ \mathsf{inc}_{A\to A} = \mathsf{inc}_{B\to B} \circ f$.

(c) **[3 Marks]** If $f : A \to B$ is bijective, then $f \circ f^{-1} = \mathsf{inc}_{B\to B}$ and $f^{-1} \circ f = \mathsf{inc}_{A\to A}$.

(d) [**3 Marks**] If X and Y are disjoint sets and $f : X \to Z$, $g : Y \to Z$ are functions, then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \mathsf{inc}_{X \to X \cup Y} = f$ and $h \circ \mathsf{inc}_{Y \to X \cup Y} = g$.

Rationale.

Equality of functions has three components: the same domain, the same range, and the same value at every point of the domain. The range is carried data, not something recovered from the values — $\mathsf{inc}_{X \to Y}$ and $\mathsf{inc}_{X \to Z}$ agree in domain and at every point, yet are different functions — and this is the whole reason the subscripts are written out. Notice also that $X \subseteq Y$ is needed for $\mathsf{inc}_{X \to Y}$ to exist at all, since the property $y = x$ must pick out exactly one y in Y. Parts (a)–(c) are then value-chasing. Part (d) is of a different kind: a function must be built, which here means exhibiting a property and checking that it determines exactly one value at each point of $X \cup Y$. Disjointness is needed.

A remark about terminology. For a function $f : X \to Y$, we call X as domain, Y as range. In many other texts Y is called codomain and range is then the image $f(X)$ of the set X under f.