Thursday, 4 September 2025 10:02 PM

Answer 1: Given $a_0 = 1$, $a_{n+1} = 2a_n + 1$

(a) $a_0 = 1$, $a_1 = 3$, $a_2 = 7$,

$a_3 = 15$, $a_4 = 31$

b) $a_n = 2^{n+1} - 1$ (Guess)

Sanity checks: $a_0 = 1$, $a_1 = 3$, $a_2 = 7 \cdots$

(c) Let us apply induction:

Assume $a_n = 2^{n+1} - 1$ is true for n.

(a) Base case: $a_0 = 2^1 - 1 = 1$ (By definition satisfied)

(b) $a_{n+1} = 2a_n + 1$

$= 2(2^{n+1} - 1) + 1$

$= 2^{n+2} - 1$ [This implies formula

[is true for n+1]

$\Rightarrow$ From induction, we have that

$a_n = 2^{n+1} - 1$ is true for all natural

numbers.

Answer 2: Existence:

Let $b \geq 2$ be a fixed natural number.

Then from division algorithm, for each $n \in N$

there exist natural numbers q_0 and a_0 with

$0 \leq a_0 < b$ such that

$$n = q_0 b + a_0$$

If $q_0 = 0$, we are done and $n = a_0$. If not then

we have $n = q_0 b + a_0$. Since $q_0 \neq 0$, we can

apply division algorithm again to get

$$q_0 = q_1 b + a_1 \quad [0 \leq a_1 < b]$$

If $q_1 = 0$, then we are done and

$$n = a_1 b + a_0 .$$

Else we have

$$n = q_1 b^2 + a_1 b + a_0$$

This process will end somewhere as naturals are bounded from below by zero.

Let the process ends in $(k+1)$ steps then

$$n = a_0 + a_1 b + \cdots + a_k b^k \quad , \text{here } a_k \neq 0.$$
and $0 \leq a_i < b \ \forall i.$

Uniqueness:

Suppose $n = a_0 + a_1 b + \cdots + a_k b^k$

and $n = c_0 + c_1 b + \cdots + c_\ell b^\ell$

where $0 \leq a_i, c_i < b.$

Assume $k \leq \ell$ (without loss of generality)

Then $0 = (a_0 - c_0) + \cdots + (a_k - c_k) b^k - (c_{k+1} b^{k+1} + \cdots + c_l b^l)$

This expresses 0 as a nontrivial base b expansion.

It is impossible unless $a_i = c_i \ \forall\ i \le k$
and $c_i = 0 \ \forall\ i \geq k+1$.

$\Rightarrow$ Uniqueness of base b expansion.

Thursday, 4 September 2025 9:58 PM

Answer 3: Given $f: \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x^k$ for $k \geq 2$.

(a) k is a natural number. Then it can either be even or odd.

Case 1 Let k is even. Then note that

$$f(1) = 1 = f(-1)$$

$\Rightarrow$ f is not injective.

Case 2 Let k is odd. Then $k = 2m+1$ for some integer $m > 0$ (as $k \geq 2$)

Suppose $x, y \in \mathbb{Z}$ and $f(x) = f(y)$. Then

If $x = y$, then function is injective and we are done.

Let $x \neq y$. Then either $x > y$ or $x < y$.

Case 1: $x > y$. Put $h = x - y > 0$. Then

$$x^k - y^k = (y+h)^k - y^k$$
$$= \sum_{i=1}^{k} \binom{k}{i} y^{k-i} h^i$$

(Case 1a): If $y \geq 0$ then each term

is summation is ≥ 0 and at least

one is >0. (e.g. $i=k$)

$\Rightarrow$ $x^k - y^k > 0$ Contradicting the assumption $x^k = y^k$.

$\Rightarrow$ x cannot be greater than y.

(Case 1b): If $y<0$, put $u=-y$ and $v=-x$

Then $x>y \Rightarrow u>v$.

Since k is odd

$$x^k - y^k = (-v)^k - (-u)^k$$
$$= u^k - v^k$$

Now since $u>y \Rightarrow u^k - v^k > 0$

Again contradicting $x^k = y^k$.

$\Rightarrow$ x can not be greater than y.

Case 2: $x<y$, Reversing the roles of x, y

in case 1, we again get that x cannot

be smaller than y.

$\Rightarrow$ $x=y$.

Thus $f(x) = f(y) \Rightarrow x=y$ when k is odd.

$\Rightarrow$ f is injective function only when k

is odd.

(b) For any $k \geqslant 2$, not every integer is a perfect k-th power. (e.g. 2 is not power of any integer for any $k \geqslant 2$)

$\Rightarrow f$ is not surjective.

Hence f is never bijective.

(c) $k=2$, then $f(x) = x^2$

Inverse image of $\mathbb{N} = \{0, 1, 2 \ldots\}$

$$f^{-1}(\mathbb{N}) = \{x \in \mathbb{Z} : x^2 \in \mathbb{N}\}$$

$$= \mathbb{Z}$$

(since square of any integer is a natural number)

Thursday, 4 September 2025 9:59 PM

Answer 4: Given for each rational x

$$d(x) = \min_{m \in \mathbb{Z}} |x-m|$$

(a) By definition of absolute value $|x-m| \geq 0$

$\Rightarrow \quad d(x) \geq 0 \quad \forall\ x \in \mathbb{Q}$

We know that for a rational x, there exists $N \in \mathbb{Z}$ s.t

$$N \leq x < N+1$$

Then $d(x) = \min \{ x-N, \ N+1-x \}$

Notice that $(x-N) + (N+1-x) = 1$

Then $\max_{x \in \mathbb{Q}} d(x) = 1/2$

$\Rightarrow \quad 0 \leq d(x) \leq 1/2.$

(b) Any rational of the form $x = m + 1/2$, where $m \in \mathbb{Z}$

gives $d(x) = 1/2$.

(c) If $x \in \mathbb{Z}$ then

$$d(x) = \min_{m \in \mathbb{Z}} |x-m|$$

Since $d(x) \geq 0$, the minimum above

is achieved at $m = x$.

$\Rightarrow \; d(x)=0$ when $x \in \mathbb{Z}$.

If $d(x)=0$ for some $x \in \mathbb{Q}$. Then

$$d(x) = \min\{x-N, \; N+1-x\}$$

But $\min\{x-N, N+1-x\} = 0$ only if either

$$x = N \quad \text{or} \quad x = N+1.$$

That only if x is an integer.

$\Rightarrow \; d(x)=0$ iff $x \in \mathbb{Z}$.

<u>Answer 5</u>: Claim: No surjection from $f: \mathbb{N} \to P(\mathbb{N})$

<u>Proof</u>: Suppose for contradiction that such a surjection exists. This means that for each $A \subseteq \mathbb{N}$, there exists some $n \in \mathbb{N}$ s.t.

$$f(n) = A.$$

Construct the set

$$S = \{n \in \mathbb{N} : n \notin f(n)\}$$

Note that S is a subset of $\mathbb{N}$ by construction. That is $S \subseteq \mathbb{N}$. Then there is some $k \in \mathbb{N}$ s.t.

$$f(k) = S. \quad [\text{As } f \text{ is surjective}]$$

Now ask if $k \in S$?

① If $k \in S$. Then by definition of S,

$$k \notin f(k) \iff k \notin S.$$

[A contradiction]

② If $k \notin S$. Then again $k \notin f(k)$ as $f(k) = S$.

Then by definition of S, $k \in S$.

[A contradiction]

$\Rightarrow$ A surjective $f : \mathbb{N} \to P(\mathbb{N})$ cannot exist.

$$\Rightarrow \quad |\mathbb{N}| < |P(\mathbb{N})|$$