International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA4.101-Real Analysis (Monsoon-2025)

Quiz 2

Time: 45 Minutes Total Marks: 20

Question (2). [10 Marks] Answer following question.

(a) [**4 Marks**] Prove or disprove that the series $\sum_{n=1}^{\infty} \frac{(n+1)^{1/n}}{n^2}$ is convergent.

(b) [**6 Marks**] Prove or disprove that the series $\sum_{n=1}^{\infty} a_n, \quad a_n = \frac{1}{2^n}\left(1 + \frac{(-1)^n}{n}\right)$ is convergent. If convergent, compute the sum of the series.

Question 4. [10 Marks] Consider the series

$$\sum_{n=3}^{\infty} \frac{1}{(n+2)\big(n+4(-1)^n\big)}.$$

Show that the series is convergent/divergent. If it is convergent, then compute the sum. Can the series be conditionally convergent?

Solutions.

Answer 2(a). For convenience set

$$a_n := \frac{(n+1)^{1/n}}{n^2} \qquad (n \geq 1). \tag{1}$$

Method 1 (Direct comparison). It suffices to show that for all sufficiently large n one has

$$(n+1)^{1/n} \leq 2,$$

because then $a_n \le 2/n^2$ and $\sum 2/n^2$ converges. To prove $(n+1)^{1/n} \le 2$ for $n \ge 2$ it is equivalent to show

$$n + 1 \le 2^n \quad \text{for } n \ge 1.$$

This inequality is standard and follows by induction. For $n = 1$ we have $2 \le 2$. Assume $n + 1 \le 2^n$ for some $n \ge 2$. Then

$$n + 2 = (n + 1) + 1 \le 2^n + 1 \le 2^n + 2^{n-1} = \tfrac{3}{2} \cdot 2^n < 2 \cdot 2^n = 2^{n+1},$$

so $n+2 \le 2^{n+1}$. Hence the induction holds and $n+1 \le 2^n$ for all $n \ge 1$. Taking n-th roots yields $(n+1)^{1/n} \le 2$ for $n \ge 1$.

Therefore for $n \ge 1$,

$$a_n = \frac{(n+1)^{1/n}}{n^2} \le \frac{2}{n^2},$$

and since $\sum_{n=2}^{\infty} \frac{2}{n^2}$ converges (a p-series with $p = 2$), the comparison test implies $\sum_{n=1}^{\infty} a_n$ converges (the first term a_1 is finite and does not affect convergence).

Method 2 (Limit-comparison test). Compare a_n with the convergent p-series $b_n := 1/n^2$. We compute

$$\frac{a_n}{b_n} = (n+1)^{1/n}.$$

We show $(n+1)^{1/n} \to 1$ as $n \to \infty$. Thus

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \in (0, \infty).$$

By the limit-comparison test, $\sum a_n$ converges if and only if $\sum b_n$ converges. Since $\sum 1/n^2$ converges, so does $\sum a_n$.

Answer 2(b). Split the series to get

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{2^n} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}.$$

Convergence. The first sum is a geometric series with ratio $1/2 < 1$, so it converges. The second sum is an alternating series with terms $|(-1)^n/(n2^n)|$ decreasing to zero, so it converges by the alternating series test. Hence, the original series converges.

Sum. First sum:

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Second sum: use the series expansion $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}x^n/n$, for $|x| < 1$, with $x = -\frac{1}{2}$:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(\frac{1}{2}\right)^n = -\ln\left(1 - \frac{1}{2}\right) = \ln 2$$

Therefore, the sum of the series is

$$\sum_{n=1}^{\infty} a_n = 1 + \ln 2.$$

Answer 4. For large n,

$$|n + (-1)^n 4| \geq n - 4,$$

so

$$\left|\frac{1}{(n+2)\big(n+(-1)^n 4\big)}\right| \leq \frac{1}{(n+2)(n-4)} \leq \frac{C}{n^2}$$

for some constant $C > 0$. Since the series $\sum 1/n^2$ converges, the given series converges absolutely (and hence converges). It can't be conditionally convergent as it is absolutely convergent.

Sum. Because of the term $(-1)^n$, we separate the series into even and odd parts:

$$S = \sum_{\substack{n=3 \\ n \text{ odd}}}^{\infty} \frac{1}{(n+2)(n-4)} + \sum_{\substack{n=3 \\ n \text{ even}}}^{\infty} \frac{1}{(n+2)(n+4)}.$$

(a) Even terms. Let $n = 2k$ $(k \geq 2)$. Then

$$a_{2k} = \frac{1}{(2k+2)(2k+4)} = \frac{1}{4(k+1)(k+2)}.$$

Since

$$\frac{1}{(k+1)(k+2)} = \frac{1}{k+1} - \frac{1}{k+2},$$

we have

$$a_{2k} = \frac{1}{4}\left(\frac{1}{k+1} - \frac{1}{k+2}\right).$$

Thus

$$S_{\text{even}} = \sum_{k=2}^{\infty} a_{2k} = \frac{1}{4}\sum_{k=2}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+2}\right).$$

This telescopes:

$$\sum_{k=2}^{K}\left(\frac{1}{k+1}-\frac{1}{k+2}\right)=\frac{1}{3}-\frac{1}{K+2}\xrightarrow[K\to\infty]{}\frac{1}{3}.$$

Hence

$$\boxed{S_{\text{even}}=\frac{1}{12}}.$$

(b) Odd terms. Let $n=2k+1$ ($k\geq 1$). Then

$$a_{2k+1}=\frac{1}{(2k+3)(2k-3)}.$$

We decompose:

$$\frac{1}{(2k+3)(2k-3)}=\frac{A}{2k-3}+\frac{B}{2k+3}.$$

Solving $A(2k+3)+B(2k-3)=1$ gives $B=-A$ and $A=\frac{1}{6}$, so

$$\frac{1}{(2k+3)(2k-3)}=\frac{1}{6}\left(\frac{1}{2k-3}-\frac{1}{2k+3}\right).$$

Therefore,

$$S_{\text{odd}}=\frac{1}{6}\sum_{k=1}^{\infty}\left(\frac{1}{2k-3}-\frac{1}{2k+3}\right).$$

Let $b_k=\frac{1}{2k-3}$. Then

$$\sum_{k=1}^{K}(b_k-b_{k+3})=b_1+b_2+b_3-(b_{K+1}+b_{K+2}+b_{K+3})\xrightarrow[K\to\infty]{}b_1+b_2+b_3.$$

Compute:

$$b_1+b_2+b_3=\frac{1}{-1}+\frac{1}{1}+\frac{1}{3}=-1+1+\frac{1}{3}=\frac{1}{3}.$$

Hence

$$\boxed{S_{\text{odd}}=\frac{1}{18}}.$$

Combining, we get

$$S=S_{\text{even}}+S_{\text{odd}}=\frac{1}{12}+\frac{1}{18}=\frac{5}{36}.$$