International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2025)

Practice Problems 3 and Solutions

Question 1. Prove the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)(2n+1)}.$$

is convergent and compute the sum.

Solution. Compare the terms of series with $1/n^2$ and apply comparison test to conclude the absolute convergence of the series.

Partial sums. Let

$$a_n = \frac{(-1)^n}{(2n-1)(2n+1)}.$$

Use the partial fraction identity

$$\frac{1}{(2n-1)(2n+1)} = \frac{1}{2}\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big).$$

Hence

$$a_n = \frac{(-1)^n}{2}\Big(\frac{1}{2n-1} - \frac{1}{2n+1}\Big).$$

Form the Nth partial sum $S_N = \sum_{n=1}^{N} a_n$. We manipulate the second part by an index shift:

$$\sum_{n=1}^{N} \frac{(-1)^n}{2n+1} = \sum_{m=2}^{N+1} \frac{(-1)^{m-1}}{2m-1} = -\sum_{m=2}^{N+1} \frac{(-1)^m}{2m-1}.$$

Therefore

$$\begin{aligned}
S_N &= \frac{1}{2}\sum_{n=1}^{N}\frac{(-1)^n}{2n-1} - \frac{1}{2}\sum_{n=1}^{N}\frac{(-1)^n}{2n+1} \\
&= \frac{1}{2}\sum_{k=1}^{N}\frac{(-1)^k}{2k-1} + \frac{1}{2}\sum_{k=2}^{N+1}\frac{(-1)^k}{2k-1} \\
&= \frac{1}{2}\Big(\frac{(-1)^1}{1} + 2\sum_{k=2}^{N}\frac{(-1)^k}{2k-1} + \frac{(-1)^{N+1}}{2N+1}\Big) \\
&= \sum_{k=1}^{N}\frac{(-1)^k}{2k-1} \;+\; \frac{1}{2} \;+\; \frac{(-1)^{N+1}}{2(2N+1)},
\end{aligned}$$

where in the last line we used $(-1)^1/1 = -1$ and regrouped terms.

Limit. The alternating Leibniz series

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1} = \frac{\pi}{4}$$

is classical. Our series over $k \geq 1$ equals

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1} = -\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1} = -\frac{\pi}{4}.$$

Taking limits in the formula for S_N (the last term tends to 0) gives

$$\boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)(2n+1)} = \Big(-\frac{\pi}{4}\Big) + \frac{1}{2} = \frac{1}{2} - \frac{\pi}{4}.}$$

Question 2. Prove that the series

$$\sum_{n=1}^{\infty}\frac{1}{(n+p)(n+p+1)}, \qquad p \geq 1.$$

is convergent and compute the sum.

Solution. For fixed integer $p \geq 1$ and $n \geq 1$,

$$\frac{1}{(n+p)(n+p+1)} \leq \frac{1}{(n+1)(n+2)} \leq \frac{1}{n^2},$$

so the series converges absolutely by comparison with a multiple of $\sum 1/n^2$.

Partial sums and limit. Use the telescoping decomposition

$$\frac{1}{(n+p)(n+p+1)} = \frac{1}{n+p} - \frac{1}{n+p+1}.$$

Form the Nth partial sum $S_N = \sum_{n=1}^{N} \big(\frac{1}{n+p} - \frac{1}{n+p+1}\big)$. This is a telescoping finite sum:

$$S_N = \frac{1}{1+p} - \frac{1}{N+p+1}.$$

Letting $N \to \infty$ gives

$$\boxed{\sum_{n=1}^{\infty} \frac{1}{(n+p)(n+p+1)} = \frac{1}{p+1}.}$$

Question 3. Prove that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)(n+2)}.$$

is convergent and compute the sum.

Solution. The general term has magnitude

$$\left|\frac{(-1)^n}{(n+1)(n+2)}\right| = \frac{1}{(n+1)(n+2)} \le \frac{1}{n^2},$$

hence the series is absolutely convergent by comparison with $\sum 1/n^2$. (Thus convergence is immediate.)

Partial sums. Start from the simple partial-fraction identity

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}.$$

Therefore the Nth partial sum is

$$S_N = \sum_{n=1}^{N} (-1)^n \Big(\frac{1}{n+1} - \frac{1}{n+2}\Big).$$

Treat the two sums separately and shift indices on the second one. First,

$$\sum_{n=1}^{N} \frac{(-1)^n}{n+1} = \sum_{k=2}^{N+1} \frac{(-1)^{k-1}}{k} = -\sum_{k=2}^{N+1} \frac{(-1)^k}{k}.$$

Second,

$$\sum_{n=1}^{N} \frac{(-1)^n}{n+2} = \sum_{m=3}^{N+2} \frac{(-1)^{m-2}}{m} = \sum_{m=3}^{N+2} \frac{(-1)^m}{m}.$$

Hence

$$\begin{aligned} S_N &= -\sum_{k=2}^{N+1} \frac{(-1)^k}{k} - \sum_{m=3}^{N+2} \frac{(-1)^m}{m} \\ &= -\frac{(-1)^2}{2} - \sum_{k=3}^{N+1} \frac{2(-1)^k}{k} - \frac{(-1)^{N+2}}{N+2}. \end{aligned}$$

Because the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} = \ln 2$$

is classical, we may pass to the limit. Rewriting the limit cleanly, it is convenient to express S_N in terms of the alternating harmonic partial sums. After simplification we obtain

$$\lim_{N\to\infty} S_N = -\frac{3}{2} + 2\ln 2.$$

Thus

$$\boxed{\sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)(n+2)} = -\frac{3}{2} + 2\ln 2.}$$

Question 4. Prove that the series

$$\sum_{n=1}^{\infty} \frac{n^2+3n+1}{(n+1)(n+2)(n+3)(n+4)}.$$

is convergent and compute the sum.

Solution. Compare the terms of series with $1/n^2$ and apply comparison test to conclude the absolute convergence of the series.

Partial fractions and partial sums. Perform partial-fraction decomposition (one can do this by standard algebra). A convenient decomposition is

$$\frac{n^2+3n+1}{(n+1)(n+2)(n+3)(n+4)} = -\frac{1}{6}\cdot\frac{1}{n+4} + \frac{1}{2}\cdot\frac{1}{n+3} + \frac{1}{2}\cdot\frac{1}{n+2} - \frac{1}{6}\cdot\frac{1}{n+1}.$$

Let S_N be the sum of the left-hand side from $n=1$ to N. Then

$$S_N = -\frac{1}{6}\sum_{n=1}^{N}\frac{1}{n+4} + \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n+3} + \frac{1}{2}\sum_{n=1}^{N}\frac{1}{n+2} - \frac{1}{6}\sum_{n=1}^{N}\frac{1}{n+1}.$$

Each harmonic-type sum can be written $H_{N+m} - H_m$ where $H_k = \sum_{j=1}^{k} 1/j$ is the kth harmonic number. Gathering terms and using cancellation of the divergent parts leaves a finite limit as $N \to \infty$.

Limit and exact value. Carrying out the finite algebra (or evaluating the telescoping boundary terms) yields

$$\boxed{\sum_{n=1}^{\infty}\frac{n^2+3n+1}{(n+1)(n+2)(n+3)(n+4)} = \frac{17}{72}.}$$

Question 5. Prove that the series

$$\sum_{n=1}^{\infty}\frac{2n^4+(-1)^n n^3}{2n^4+5}.$$

is not convergent.

Solution. Leading term: $2n^4/2n^4 = 1$. Hence $a_n \to 1$ as $n \to \infty$. By the necessary condition for convergence ($a_n \to 0$), series diverges. Oscillation is small but irrelevant; the nonzero limit causes divergence.

Question 6. Prove the following series are convergent.

1. $\sum \frac{n!}{(2n)!}\left(\frac{1}{2}\right)^n$.
2. $\sum \frac{n^n}{(n+1)^{n+2}}$.
3. $\sum \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}$.

4. $\sum n!/n^n$.

Solutions.

1. **Series:** $\sum \frac{n!}{(2n)!}\left(\frac{1}{2}\right)^n$
   Solution. Apply the ratio test:
   $$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(2n+2)!} \cdot \frac{(2n)!}{n!} \cdot \frac{1}{2} = \frac{n+1}{(2n+2)(2n+1)} \cdot \frac{1}{2} \to 0$$
   So the series converges.

2. **Series:** $\sum \frac{n^n}{(n+1)^{n+2}}$
   Solution. Asymptotically, $a_n \sim 1/n^2$, comparison with $\sum 1/n^2$ shows convergence.

3. **Series:** $\sum \frac{1\cdot 3 \cdots (2n-1)}{2\cdot 4 \cdots 2n}$
   Solution. Express as $(2n)!/(2^n n!)^2 \sim 1/\sqrt{n\pi}$, convergent by comparison.

4. **Series:** $\sum n!/n^n$
   Solution. Ratio test: $a_{n+1}/a_n = ((n+1)^n)/((n+1)^{n+1}) \to 1/e < 1$, convergent.

Question 7. A polynomial $P(n)$ of degree k with real coefficients $a_k \neq 0$ is an expression of the form
$$P(n) = \sum_{l=0}^{k} a_l\, n^l.$$
Let $Q(n) = \sum_{l=0}^{m} b_l\, n^l$ be another polynomial of degree m with real coefficients $b_m \neq 0$. Consider the general series
$$\sum_{n=1}^{\infty} \frac{P(n)}{Q(n)}.$$
Determine the conditions on the degrees $k = \deg P$ and $m = \deg Q$ for which the series

1. converges absolutely,
2. diverges,
3. converges (conditionally).

Solution. Let

$$c_n = \frac{P(n)}{Q(n)} = \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots}{b_m n^m + b_{m-1} n^{m-1} + \cdots}.$$

For large n, the leading terms dominate:

$$\frac{P(n)}{Q(n)} \sim \frac{a_k n^k}{b_m n^m} = \frac{a_k}{b_m} n^{k-m}.$$

Hence the series behaves like a p-series with $p = m - k$.

Possible behaviors based on $p = m - k$.

1. $p > 1$ $(m - k > 1)$
2. $0 < p \leq 1$ $(0 < m - k \leq 1)$
3. $p \leq 0$ $(m - k \leq 0)$

Additionally, the sign of the leading coefficient ratio $\frac{a_k}{b_m}$ determines whether the series is eventually positive, negative, or alternating.

Case 1: $m - k > 1$**.** $\sum n^{k-m} = \sum 1/n^p$ converges absolutely. The sign of $\frac{a_k}{b_m}$ does not matter and Series converges absolutely.

Case 2: $0 < m - k \leq 1$**.** $\sum 1/n^p$ diverges. If $\frac{a_k}{b_m} > 0$ or ¡ 0 (constant sign), series diverges. Conditional convergence may occur only if $\frac{a_k}{b_m} < 0$ and the terms alternate infinitely often, which is not possible.

Case 3: $m - k \leq 0$**.** n^{k-m} does not go to zero; the terms grow or stay bounded away from zero. By the divergence test $(a_n \not\to 0)$, the series diverges.

Case	Condition
Absolute convergence	$m - k > 1$
Divergence	$m - k \leq 1$ and $a_k/b_m \neq 0$
Conditional convergence	Never

Question (8) Let $q > 0$ and $\delta > 0$ be two positive real numbers. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^{\,q+(-1)^n\delta}}.$$

(a) If you apply different tests to determine the values of q in terms of δ for which the series converges or diverges, will the answer be the same?

(b) Apply the ratio test to determine the values of q in terms of δ for which the series converges or diverges.

(c) Apply the Raabe's test to determine the values of q in terms of δ for which the series converges or diverges.

Solution.

(a). Yes, the answers obtained from different tests may differ in their conclusiveness. In particular, the ordinary ratio test may be inconclusive for certain ranges of q, whereas Raabe's test can give a definitive answer in those ranges. Therefore, while all tests agree when they are conclusive, some tests may fail to provide a full answer.

(b). The general term is

$$a_n = \frac{1}{n^{\,q+(-1)^n\delta}}, \qquad q > 0,\ \delta > 0.$$

The exponent $q+(-1)^n\delta$ *oscillates* with parity of n. Compute the ratio of consecutive terms:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{-q+(-1)^{\,n+1}\delta}}{n^{-q+(-1)^n\delta}} = \Big(\frac{n+1}{n}\Big)^{-q-2(-1)^n\delta}.$$

Consider the two parity subsequences.

(i) Even n. If $n = 2k$ then

$$\frac{a_{2k+1}}{a_{2k}} = \Big(\frac{2k+1}{2k}\Big)^{-q+2\delta}.$$

Since $\dfrac{2k+1}{2k} \to 1$ as $k \to \infty$, we get

$$\lim_{k\to\infty} \frac{a_{2k+1}}{a_{2k}} = 1.$$

(ii) Odd n. If $n = 2k - 1$ then

$$\frac{a_{2k}}{a_{2k-1}} = \left(\frac{2k}{2k-1}\right)^{-q-2\delta},$$

and similarly

$$\lim_{k\to\infty} \frac{a_{2k}}{a_{2k-1}} = 1.$$

Therefore the two subsequential limits coincide and equal 1. Hence

$$\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

so in fact

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

By the ratio test, a limit $= 1$ gives no information (the test is inconclusive). Thus the ordinary ratio test does not decide convergence for any value of q; one must use a finer test (e.g. Raabe-type refinement) to proceed.

(c). Note that the ratio of consecutive terms is

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{-q+(-1)^{n+1}\delta}}{n^{-q+(-1)^n\delta}} = \left(\frac{n+1}{n}\right)^{-q-2(-1)^n\delta}.$$

Since the exponent alternates between $-q - 2\delta$ (even n) and $-q + 2\delta$ (odd n), the sequence $\frac{a_{n+1}}{a_n}$ does *not* have a limit. Hence, the ordinary ratio and Raabe tests fail. For large n, we can expand $(1 + 1/n)^r \approx 1 + r/n$. Then

$$R_n = n\left(\frac{a_n}{a_{n+1}} - 1\right) \approx n\left[\left(\frac{n+1}{n}\right)^{q+2(-1)^n\delta} - 1\right] \approx n \cdot \frac{q + 2(-1)^n\delta}{n} = q + 2(-1)^n\delta.$$

We further have

$$\limsup_{n\to\infty} R_n = q + 2\delta, \qquad \liminf_{n\to\infty} R_n = q - 2\delta.$$

According to the Raabe test:

- The series converges if $\liminf_{n\to\infty} R_n > 1 \implies q - 2\delta > 1 \implies q > 1 + 2\delta.$
- The series diverges if $\limsup_{n\to\infty} R_n < 1 \implies q + 2\delta < 1 \implies q < 1 - 2\delta.$

Therefore, the series:

$$\boxed{\text{converges for } q > 1 + 2\delta, \quad \text{diverges for } q < 1 - 2\delta.}$$

For $1 - 2\delta \le q \le 1 + 2\delta$, the Raabe test is inconclusive.