International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA4.101-Real Analysis (Monsoon-2025)

Practice Problems 2 and Solutions

Question (1) Answer the following questions.

(a) **Single-element set.** If X consists of a single element, $X = \{x_0\}$, and $f : X \to \mathbb{R}$ is a function, show that

$$\sum_{x \in X} f(x) = f(x_0).$$

(b) **Substitution.** Let X be a finite set, $f : X \to \mathbb{R}$ a function, and $g : Y \to X$ a bijection. Show that

$$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y)).$$

(c) **Disjoint union of finite sets.** Let X, Y be disjoint finite sets ($X \cap Y = \emptyset$), and let $f : X \cup Y \to \mathbb{R}$ be a function. Show that

$$\sum_{z \in X \cup Y} f(z) = \sum_{x \in X} f(x) + \sum_{y \in Y} f(y).$$

Solution 1

Recall that Tao defines a sum over a finite set X as follows: if $X = \{x_1, \ldots, x_N\}$ and $f : X \to \mathbb{R}$, then we choose a bijection

$$h : \{1, 2, \ldots, N\} \to X$$

and define

$$\sum_{x \in X} f(x) := \sum_{n=1}^{N} f(h(n)).$$

This definition is independent of the choice of bijection.

(a) **Single-element set.** Here $X = \{x_0\}$, so $N = 1$. Define $h(1) = x_0$. Then by definition,

$$\sum_{x \in X} f(x) = \sum_{n=1}^{1} f(h(n)) = f(h(1)) = f(x_0).$$

(b) **Substitution.** Let X be finite with $|X| = N$, and let $g : Y \to X$ be a bijection. Take a bijection $h : \{1, 2, \dots, N\} \to Y$. Then $g \circ h : \{1, 2, \dots, N\} \to X$ is a bijection, and by Tao's definition,

$$\sum_{y \in Y} f(g(y)) = \sum_{n=1}^{N} f(g(h(n))) = \sum_{x \in X} f(x),$$

since $g \circ h$ enumerates all elements of X exactly once.

(c) **Disjoint union of finite sets.** Let $|X| = N$, $|Y| = M$. Choose bijections

$$h_X : \{1, 2, \dots, N\} \to X, \quad h_Y : \{1, 2, \dots, M\} \to Y.$$

Then a bijection $h : \{1, 2, \dots, N + M\} \to X \cup Y$ can be defined by

$$h(n) = \begin{cases} h_X(n), & 1 \le n \le N, \\ h_Y(n - N), & N + 1 \le n \le N + M. \end{cases}$$

By Tao's definition,

$$\sum_{z \in X \cup Y} f(z) = \sum_{n=1}^{N+M} f(h(n)) = \sum_{x \in X} f(x) + \sum_{y \in Y} f(y).$$

Question 2 : Series with a Hidden Telescoping Structure. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}.$$

Solution 2.

Decompose using partial fractions:

$$\frac{1}{n(n+2)} = \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n+2} \right).$$

Then

$$\sum_{n=1}^{N} \frac{1}{n(n+2)} = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}\right) \to \frac{3}{4} \quad \text{as } N \to \infty.$$

Question 3: Comparison Test Application. Determine the convergence of

$$\sum_{n=1}^{\infty} b_n, \quad b_n = \frac{1}{n^2+1}.$$

Solution 3.

Since $b_n \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the comparison test, $\sum b_n$ converges.

Question 4: Conditional Convergence. Determine whether

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges absolutely, conditionally, or diverges.

Solution:

- Absolute: $\sum |(-1)^n/n| = \sum 1/n$ diverges (harmonic series).
- Conditional: By the alternating series test, $1/n$ decreases to 0, so series converges conditionally.

Question 5: Rearrangement of Terms. Rearrange the series $\sum_{n=1}^{\infty}(-1)^{n+1}/n$ to sum to 2.

Solution: By the Riemann rearrangement theorem, any conditionally convergent series can be rearranged to sum to any real number, including 2.

Question 6: Series and Integral Comparison. Compare convergence of

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

with the integral $\int_1^{\infty} \frac{dx}{x^2+1}$.

Solution:

$$\int_1^{\infty} \frac{dx}{x^2+1} = \frac{\pi}{4}.$$

Since $1/(x^2+1)$ is positive and decreasing, by the integral test, the series converges.

Question 7: Series with Non-Monotonic Terms. Determine convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

Solution:

- Absolute: $\sum 1/\sqrt{n}$ diverges.
- Conditional: By alternating series test, $\sum(-1)^n/\sqrt{n}$ converges conditionally.

Question 8: Series with Logarithmic Terms. Evaluate

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}.$$

Solution: Use integral test:

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \left[-\frac{\ln x}{x}\right]_1^{\infty} + \int_1^{\infty} \frac{dx}{x^2} = 0 + 1 = 1.$$

Hence series converges.

Question 9: Series with Exponential Terms. Determine convergence of

$$\sum_{n=1}^{\infty} \frac{e^{-n}}{n}.$$

Solution: Compare with $\sum 1/n^2$. Since $e^{-n}/n < 1/n^2$ for $n \geq 2$, series converges by comparison test.

Question 10: Series with Factorial Terms. Evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n!}.$$

Solution: Recognize as Taylor series for e^x at $x = 1$:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e \implies \sum_{n=1}^{\infty} \frac{1}{n!} = e - 1.$$

Question 11: Series with Polynomial Terms. Determine convergence of

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}.$$

Solution: For large n, $\frac{n}{n^2+1} \sim \frac{1}{n}$. Harmonic series diverges, so series diverges.

Question 12: Series with Nested Sums. Evaluate

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{n^2+m^2}.$$

Solution: Interchange sums and use known double series results:

$$\sum_{n,m=1}^{\infty} \frac{1}{n^2+m^2} = \frac{\pi}{2} \ln 2.$$

Question 13: Series with Power Terms Determine convergence of

$$\sum_{n=1}^{\infty} \frac{n^p}{n^2+1}, \quad p > 0.$$

Solution: For large n, term $\sim n^{p-2}$. Series converges if $p-2 < -1 \implies p < 1$.
Question 14: Series with Mixed Terms. Evaluate

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+n}.$$

Solution: Decompose:

$$\frac{(-1)^n}{n^2+n} = \frac{(-1)^n}{n} - \frac{(-1)^n}{n+1}.$$

This telescopes, sum equals $\ln 2$.
Question 15: Series with Logarithmic and Exponential Terms. Determine convergence of

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^n}.$$

Solution: Since n^n grows faster than $\ln n$, the general term tends to 0 extremely rapidly. By comparison with geometric series $1/2^n$, series converges.