International Institute of Information Technology, Hyderabad

(Deemed to be University)

MA4.101-Real Analysis (Monsoon-2025)

Practice Problems 1 and Solutions

Question (1) Induction isn't just for sums. It lets us compare growth rates. The fact that exponentials beat polynomials is central in analysis. Prove that for every $n \in \mathbb{N}$, we have

$$n < 2^n.$$

Question (2) Recursive definitions often hide beautiful closed formulas. This problem shows how induction transforms a recursive process into a neat algebraic expression. Define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(0) = 0, \quad f(n+1) = f(n) + (n+1).$$

Find a closed formula for $f(n)$ and prove its correctness.

Question (3) Recursive sequences often grow quickly, but how quickly? Induction lets us compare Fibonacci with exponentials, foreshadowing analysis of growth rates. Let $F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n$. Prove that

$$F_n < 2^n \quad \text{for all } n \geq 1.$$

Question (4) The rationals look "sparse," but they're so densely packed that between any two rationals lie infinitely many others. This prepares us for how rationals approximate the reals. Answer the following questions. (a) Between any two integers $a < b$, show there are infinitely many rationals. (b) Between any two rationals $p < q$, show there are infinitely many rationals.

Question (5) A set T is called countable if there is a bijective mapping from $\mathbb{N}$ to

T. Further, the union of countable sets is countable. The set T is said to be dense in another set S iff for each $a, b \in S$ with $a < b$, there exists at least one element $x \in T$ such that $a < x < b$. The rationals of the form $p/2^k$ (dyadic rationals) are countable, yet dense. They foreshadow binary expansions and approximations to real numbers. Let

$$T = \left\{ \tfrac{p}{2^k} : p \in \mathbb{Z}, k \in \mathbb{N} \right\}.$$

(a) Show that T is countable. (b) Show that T is dense in $\mathbb{Q}$.

Question (6) Prove that the set $\mathbb{Z}$ of integers is not dense in set $\mathbb{Q}$ of rationals.

Question (7) Infinity behaves differently: removing infinitely many elements may leave a set "the same size." This problem illustrates that infinite sets do not follow finite intuition. Let

$$S = \{ n \in \mathbb{N} : n \text{ is not a multiple of } 3 \}.$$

Show that S has the same cardinality as $\mathbb{N}$.

Question (8) With infinitely many pigeons and finitely many holes, at least one hole contains infinitely many pigeons. This principle underlies compactness arguments later. Let $f : \mathbb{N} \to \{1, 2, 3, 4, 5\}$. Prove that some value in $\{1, 2, 3, 4, 5\}$ is taken infinitely often by f.
[*Hint: Partition* $\mathbb{N}$ *into subsets* $S_i = \{n : f(n) = i\}$ *for* $i = 1, \ldots, 5$.]

Question (9) Telescoping and binomial bounds prepare us for convergence. This is the first glimpse of analysis of infinite processes. Let $0 < r < 1$ be rational. Prove that for each $n \geq 2$, show that

$$0 < 1 - (1 - r)^n < nr.$$

Question (10) Let $\frac{p}{q}, \frac{r}{s} \in \mathbb{Q}$ with $p, r \in \mathbb{Z}$ and $q, s > 0$. Prove that if $\frac{p}{q} \neq \frac{r}{s}$, then

$$\left| \frac{p}{q} - \frac{r}{s} \right| \geq \frac{1}{qs}.$$

Write the condition for equality in above inequality.

Solution 1

We want to prove that for every natural number n,

$$n < 2^n.$$

Step 1: Base case. If $n = 0$, the inequality says $0 < 2^0$. Since $2^0 = 1$, the inequality $0 < 1$ holds true. (If we start $\mathbb{N}$ from 1, then the base case is $n = 1$, where $1 < 2$ is true.)
Step 2: Induction hypothesis. Assume for some $k \geq 0$ that $k < 2^k$.
Step 3: Induction step. We must prove $k + 1 < 2^{k+1}$.
From the induction hypothesis $k < 2^k$, adding 1 to both sides gives

$$k + 1 < 2^k + 1.$$

Since $2^k \geq 1$, we know $2^k + 1 \leq 2^k + 2^k = 2^{k+1}$. Therefore

$$k + 1 < 2^k + 1 \leq 2^{k+1}.$$

Step 4: Conclusion. By induction, $n < 2^n$ holds for all $n \in \mathbb{N}$.

Solution 2

We are given $f(0) = 0$ and $f(n + 1) = f(n) + (n + 1)$, and must find a closed form.
Step 1: Compute first values.

$$f(0) = 0, \quad f(1) = 0 + 1 = 1, \quad f(2) = 1 + 2 = 3, \quad f(3) = 3 + 3 = 6.$$

These are the triangular numbers.
Step 2: Conjecture formula. We suspect

$$f(n) = \frac{n(n+1)}{2}.$$

Step 3: Base case. For $n = 0$, the formula gives $0(0 + 1)/2 = 0$, which matches $f(0) = 0$.
Step 4: Induction step. Assume $f(n) = n(n + 1)/2$. Then

$$f(n + 1) = f(n) + (n + 1) = \frac{n(n+1)}{2} + (n + 1).$$

Factor $(n + 1)$:

$$f(n + 1) = (n + 1)\left(\frac{n}{2} + 1\right) = (n + 1)\frac{n + 2}{2} = \frac{(n + 1)(n + 2)}{2}.$$

This matches the formula with n replaced by $n + 1$.
Step 5: Conclusion. By induction, the closed form $f(n) = n(n + 1)/2$ holds for all n.

Solution 3

We want to prove $F_n < 2^n$ for all $n \geq 1$.
Step 1: Base cases. $F_1 = 1 < 2^1 = 2$, true. $F_2 = 1 < 4$, true.
Step 2: Induction hypothesis. Suppose $F_n < 2^n$ and $F_{n+1} < 2^{n+1}$.
Step 3: Induction step.

$$F_{n+2} = F_{n+1} + F_n < 2^{n+1} + 2^n = 3 \cdot 2^n.$$

But $3 \cdot 2^n < 4 \cdot 2^n = 2^{n+2}$. Thus $F_{n+2} < 2^{n+2}$.
Step 4: Conclusion. By induction, $F_n < 2^n$ for all $n \geq 1$.

Solution 4

(a) Between integers $a < b$, consider

$$a + \frac{1}{k}, \quad k = 2, 3, 4, \ldots$$

These are rationals and lie strictly between a and $a + 1 \leq b$. Hence infinitely many.

(b) Between rationals $p < q$. Consider the mediant

$$\frac{p+q}{2}.$$

This lies strictly between p and q. Repeating with p and $\frac{p+q}{2}$ produces infinitely many rationals in between.

Solution 5

(i) $T = \bigcup_{k=0}^{\infty} \{p/2^k : p \in \mathbb{Z}\}$. Each set is countable (bijection with $\mathbb{Z}$). Countable union of countable sets is countable. So T is countable.

(ii) Density: Let $a < b$ rationals. Choose k large such that $1/2^k < b - a$. Then there exists integer p with

$$\frac{p}{2^k} \in (a, b).$$

Thus T is dense.

Solution 6

Consider the open interval $(0,1)$ in $\mathbb{R}$. The corresponding open set in $\mathbb{Q}$ is

$$(0,1) \cap \mathbb{Q} = \{\, q \in \mathbb{Q} : 0 < q < 1 \,\}.$$

This set is nonempty, for example $\frac{1}{2} \in (0,1) \cap \mathbb{Q}$. However, there is no integer $m \in \mathbb{Z}$ with $0 < m < 1$. Thus

$$(0,1) \cap \mathbb{Q} \cap \mathbb{Z} = \varnothing.$$

So we have found a nonempty open interval in $\mathbb{Q}$ that contains no element of $\mathbb{Z}$. Hence $\mathbb{Z}$ is not dense in $\mathbb{Q}$.

Solution 7

Let

$$S = \{\, n \in \mathbb{N} : n \text{ is not a multiple of } 3 \,\}, \qquad \mathbb{N} = \{0, 1, 2, 3, \dots\}.$$

We will construct an explicit bijection $f : \mathbb{N} \to S$.

Every $n \in \mathbb{N}$ is either even ($n = 2k$) or odd ($n = 2k+1$) for some $k \geq 0$. Define

$$f(n) = \begin{cases} 3k+1, & \text{if } n = 2k \text{ (even)}, \\ 3k+2, & \text{if } n = 2k+1 \text{ (odd)}. \end{cases}$$

Examples:

$$f(0) = 1, \quad f(1) = 2, \quad f(2) = 4, \quad f(3) = 5, \quad f(4) = 7, \quad f(5) = 8, \ \dots$$

This lists exactly the elements of $S = \{1, 2, 4, 5, 7, 8, \dots\}$.

Injectivity. If $n_1 \neq n_2$, then either their parities differ (so $f(n_1)$ and $f(n_2)$ lie in different residue classes modulo 3), or they have the same parity but different k, giving different outputs. Hence f is injective.

Surjectivity. Let $m \in S$. Then $m \equiv 1$ or $2 \pmod 3$.

- If $m = 3k+1$, then $m = f(2k)$.
- If $m = 3k+2$, then $m = f(2k+1)$.

Thus every $m \in S$ has a preimage under f, so f is surjective.

Therefore f is a bijection, and we conclude

$$|S| = |\mathbb{N}|.$$

Solution 8

Partition $\mathbb{N}$ into sets $S_i = \{n : f(n) = i\}$ for $i = 1, \ldots, 5$. If all S_i were finite, their union would be finite. But $\mathbb{N}$ is infinite. So at least one S_i must be infinite. Hence some value of f is taken infinitely often.

Solution 9

Set $a := 1 - r$, so that $0 < a < 1$ since $0 < r < 1$. Using the standard factorization of a geometric power, we have

$$1 - a^n = (1 - a)\big(1 + a + a^2 + \cdots + a^{n-1}\big).$$

Substituting $a = 1 - r$ gives

$$1 - (1 - r)^n = r\big(1 + (1 - r) + (1 - r)^2 + \cdots + (1 - r)^{n-1}\big).$$

Since each term $(1 - r)^k$ in the sum is positive, the sum is positive, and hence

$$1 - (1 - r)^n > 0.$$

For the upper bound, note that $(1 - r)^k < 1$ for all $k \geq 1$, so

$$1 + (1 - r) + \cdots + (1 - r)^{n-1} < 1 + 1 + \cdots + 1 = n.$$

Multiplying both sides by $r > 0$ gives

$$1 - (1 - r)^n < nr.$$

Combining the inequalities, we obtain

$$0 < 1 - (1 - r)^n < nr,$$

as required.

Solution 10

We compute the difference using a common denominator:

$$\frac{p}{q} - \frac{r}{s} = \frac{ps - rq}{qs}.$$

Here $ps - rq$ is an integer. Since $\frac{p}{q} \neq \frac{r}{s}$, the numerator $ps - rq$ is nonzero, so $|ps - rq|$ is a positive integer. Every positive integer is at least 1, hence

$$\left|\frac{p}{q} - \frac{r}{s}\right| = \frac{|ps - rq|}{qs} \geq \frac{1}{qs}.$$

This establishes the claim. $\square$

Remark. Equality can occur: if $ps - rq = \pm 1$ then

$$\left|\frac{p}{q} - \frac{r}{s}\right| = \frac{1}{qs}.$$

For instance,

$$\frac{1}{2} - \frac{1}{3} = \frac{1}{6} = \frac{1}{2 \cdot 3}.$$