International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2025)

Solution to Mid-Semester Exam

Time: 90 Minutes Total Marks: 40

Solution 2.

(a) Consider the real number $N\sqrt{2}$. By the Archimedean property, there exists an integer m such that
$$m \leq N\sqrt{2} < m + 1.$$
Dividing by N gives
$$\frac{m}{N} \leq \sqrt{2} < \frac{m+1}{N},$$
and therefore
$$0 \leq \sqrt{2} - \frac{m}{N} < \frac{1}{N}.$$

(b) This shows that for each N there is a rational m/N within distance $1/N$ of $\sqrt{2}$. Since $1/N \to 0$ as $N \to \infty$, these rationals approximate $\sqrt{2}$ arbitrarily well. Thus $\sqrt{2}$ can be approximated by rationals, and in fact the same argument works for any real number, proving that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(c) For $N = 50$, compute $50\sqrt{2} \approx 70.71$. Take $m = 70$. Then
$$\frac{70}{50} = 1.4 < \sqrt{2} < \frac{71}{50} = 1.42.$$
Hence $0 < \sqrt{2} - \frac{70}{50} < \frac{1}{50} = 0.02$. Numerically, $\sqrt{2} \approx 1.4142$, so the actual error is about 0.0142, which satisfies the bound.

Solution 4.

(a) If $x_n \to L$ with $L \neq 0$ then indeed $y_n \to 1$.

Proof: Since $x_n \to L$ we also have $x_{n+1} \to L$. Because $L \neq 0$ there exists N so that $x_{n+1} \neq 0$ for all $n \geq N$, and the quotient limit law applies:

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \frac{x_n}{x_{n+1}} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} x_{n+1}} = \frac{L}{L} = 1.$$

Thus the necessary (and sufficient for this argument) condition is that the common limit L is nonzero. If $L = 0$ no general conclusion holds (see parts (b),(c)).

(b) Example with limit -1.

Take

$$x_n = \frac{(-1)^n}{n} \qquad (n \geq 1).$$

Clearly $x_n \to 0$. Compute

$$y_n = \frac{x_n}{x_{n+1}} = \frac{(-1)^n/n}{(-1)^{n+1}/(n+1)} = -\frac{n+1}{n} \longrightarrow -1 \quad (n \to \infty).$$

Hence $x_n \to 0$ but $y_n \to -1$.

(c) Example where $x_n \to 0$ but y_n has no finite limit (indeed is unbounded).

Define

$$x_n = \begin{cases} \dfrac{1}{n}, & n \text{ odd}, \\ \dfrac{1}{\sqrt{n}}, & n \text{ even}. \end{cases}$$

Both subsequences $1/n$ and $1/\sqrt{n}$ tend to 0, so $x_n \to 0$. Now examine $y_n = x_n/x_{n+1}$.

If n is odd (write $n = 2k - 1$) then

$$y_n = \frac{1/n}{1/\sqrt{n+1}} = \frac{\sqrt{n+1}}{n} = \frac{\sqrt{2k}}{2k-1} \sim \frac{1}{\sqrt{n}} \to 0.$$

If n is even (write $n = 2k$) then

$$y_n = \frac{1/\sqrt{n}}{1/(n+1)} = \frac{n+1}{\sqrt{n}} \sim \sqrt{n} \to +\infty.$$

Thus the even-indexed subsequence (y_{2k}) is unbounded (tends to $+\infty$), while the odd-indexed subsequence $(y_{2k-1}) \to 0$. Therefore (y_n) does not converge to any finite value.

(Any other construction with alternating scales that produce a small/large ratio will serve.)

(d) If (x_n) is monotone (increasing or decreasing) and $x_n \to L \neq 0$, the conclusion does not change: we still have $y_n \to 1$. Monotonicity is *not* needed for the positive result in (a); the crucial hypothesis was $L \neq 0$ so that the denominator sequence x_{n+1} converges to a nonzero limit and the quotient law applies. Monotonicity merely gives extra regularity (for instance it guarantees eventual sign-stability), but it is not required for $\lim y_n = 1$.

Solution 5

We are given the sequence $(a_n)_{n\geq 1}$ with

$$a_n = \frac{(-1)^n}{2}.$$

Thus, the sequence alternates between $-\frac{1}{2}$ and $+\frac{1}{2}$.

(a) [**1 Mark**] The sequence oscillates between two distinct values $-\frac{1}{2}$ and $+\frac{1}{2}$ and hence does not converge. Its set of limit points is $\{-\frac{1}{2}, +\frac{1}{2}\}$.

(b) [**5 Marks**] Let $L \geq \frac{1}{2}$ be a positive rational number. Define

$$d_n = \inf_{k\geq n} |a_k - L|.$$

We compute:

$$|a_k - L| = \begin{cases} L - \frac{1}{2}, & a_k = \frac{1}{2}, \\ L + \frac{1}{2}, & a_k = -\frac{1}{2}. \end{cases}$$

Thus the sequence $\{|a_k - L|\}_{k\geq 1}$ alternates between $L - \frac{1}{2}$ and $L + \frac{1}{2}$. Since both values appear infinitely often in every tail, we obtain

$$d_n = \min\{L - \tfrac{1}{2},\, L + \tfrac{1}{2}\} = L - \tfrac{1}{2}, \quad \forall n.$$

Hence the sequence (d_n) is constant, so

$$\lim_{n\to\infty} d_n = L - \tfrac{1}{2}.$$

The set of limit points of (d_n) is the singleton $\{L - \frac{1}{2}\}$.

(c) [**4 Marks**] Define

$$e_n = \min_{1\leq k\leq n} |a_k - L|.$$

Again, the values are $L - \frac{1}{2}$ and $L + \frac{1}{2}$. - For $n = 1$, we have $a_1 = -\frac{1}{2}$, hence

$$e_1 = |a_1 - L| = L + \tfrac{1}{2}.$$

- For $n \geq 2$, both $a_k = \pm\frac{1}{2}$ occur among the first n terms, so

$$e_n = \min\{L - \tfrac{1}{2}, L + \tfrac{1}{2}\} = L - \tfrac{1}{2}.$$

Therefore

$$e_n = \begin{cases} L + \frac{1}{2}, & n = 1, \\ L - \frac{1}{2}, & n \geq 2. \end{cases}$$

Hence the sequence (e_n) converges and

$$\lim_{n\to\infty} e_n = L - \tfrac{1}{2}.$$

Solution 6.

(a) *First six terms.* We check which of $2, 3, 4, 5, 6$ are factorials: $2! = 2$ and $3! = 6$. Thus

$$\begin{aligned} a_1 &= 4, \\ a_2 &= -1 + \tfrac{1}{2} = -\tfrac{1}{2}, \\ a_3 &= 1 - \tfrac{1}{3} = \tfrac{2}{3}, \\ a_4 &= 1 - \tfrac{1}{4} = \tfrac{3}{4}, \\ a_5 &= 1 - \tfrac{1}{5} = \tfrac{4}{5}, \\ a_6 &= -1 + \tfrac{1}{3} = -\tfrac{2}{3}. \end{aligned}$$

(b) *Boundedness.* For $n \geq 2$ there are two cases:

- If $n = k!$ for some $k \geq 2$, then $a_n = -1 + \dfrac{1}{k}$. Since $k \geq 2$ we have $-1 + \dfrac{1}{k} > -1$ and in fact $-1 + \dfrac{1}{k} \geq -\frac{1}{2}$ for $k \geq 2$.

- If n is not a factorial, then $a_n = 1 - \dfrac{1}{n}$, so $a_n < 1$ and for $n \geq 2$ we have $a_n \geq \frac{1}{2}$.

Also $a_1 = 4$. Hence for every $n \geq 1$ we have

$$-1 < a_n \leq 4.$$

Therefore the sequence is bounded below (take lower bound -1) and bounded above (take upper bound 4).

(c) *Supremum and infimum.*

Supremum. Since $a_1 = 4$ and every other term satisfies $a_n < 4$, the number 4 is an upper bound of the set $\{a_n : n \geq 1\}$ and it is attained at $n = 1$. No larger number below 4 can be an upper bound, so

$$\sup_{n \geq 1} a_n = 4,$$

and this supremum is attained (indeed it is the maximum).

Infimum. The factorial subsequence $a_{k!} = -1 + \dfrac{1}{k}$ satisfies

$$-1 < a_{k!} < 0 \quad \text{for all } k \geq 2,$$

and $\lim_{k\to\infty} a_{k!} = -1$. Thus -1 is a lower bound of the set $\{a_n : n \geq 1\}$. To see that it is the greatest lower bound, let $\varepsilon > 0$ be arbitrary. Choose k so large that $\dfrac{1}{k} < \varepsilon$. Then $a_{k!} = -1 + \dfrac{1}{k} < -1 + \varepsilon$, showing that no number > -1 can serve as a lower bound. Therefore

$$\inf_{n \geq 1} a_n = -1.$$

The value -1 is *not* attained by any a_n (every $a_{k!} > -1$ and other terms are > 0), so the infimum is not a minimum.

(d) *Computation of* $\limsup$ *and* $\liminf$. We use Tao's tail-sup and tail-inf construction:

$$a_N^+ := \sup\{a_n : n \geq N\}, \qquad a_N^- := \inf\{a_n : n \geq N\},$$

and then

$$\limsup_{n\to\infty} a_n = \lim_{N\to\infty} a_N^+, \qquad \liminf_{n\to\infty} a_n = \lim_{N\to\infty} a_N^-.$$

Compute a_N^+. For $N = 1$ we have $a_1 = 4$ so $a_1^+ = 4$. For any $N \geq 2$, the tail $\{a_n : n \geq N\}$ contains infinitely many non-factorial indices n with $a_n = 1 - \dfrac{1}{n}$, and these values approach 1 from below as $n \to \infty$. Also the factorial entries in the tail are ≤ 0. Hence for every $N \geq 2$ the supremum of the tail is the limit point 1 (the supremum of values arbitrarily close to 1 from below is 1). Thus

$$a_N^+ = \begin{cases} 4, & N = 1, \\ 1, & N \geq 2. \end{cases}$$

Therefore $\lim_{N\to\infty} a_N^+ = 1$, and so

$$\boxed{\limsup_{n\to\infty} a_n = 1.}$$

(One can justify the claim "$a_N^+ = 1$ for $N \geq 2$" by: for any $\delta > 0$ choose m large and not a factorial with $m \geq N$ so that $1 - \frac{1}{m} > 1 - \delta$; hence $\sup_{n \geq N} a_n \geq 1 - \delta$ for every $\delta > 0$, which forces the supremum to be ≥ 1, while no tail element exceeds 1, so the supremum equals 1.)

Compute a_N^-. For any $N \geq 1$ the tail $\{a_n : n \geq N\}$ contains factorial indices $k!$ arbitrarily large (factorials tend to infinity), and at such indices $a_{k!} = -1 + \dfrac{1}{k}$ can be made arbitrarily close to -1 from above. Hence for every N the infimum of the tail is -1. Thus

$$a_N^- = -1 \quad \text{for all } N \geq 1,$$

and so $\lim_{N\to\infty} a_N^- = -1$. Therefore

$$\boxed{\liminf_{n\to\infty} a_n = -1.}$$

Solution 7.

(a) *Every monotone increasing sequence is quasi-monotone increasing.*

If (x_n) is monotone increasing then $x_{n+1} \geq x_n$ for every n, hence for every n

$$x_{n+1} \geq x_n \geq x_n - \frac{1}{6}.$$

Thus the quasi-monotone inequality holds with $N = 1$. $\square$

(b) *Show (x_n) and (y_n) are quasi-monotone increasing but not monotone.*

(i) The sequence $x_n = 1 + \dfrac{(-1)^n}{18n}$.

Not monotone: the terms alternate around 1. For example

$$x_1 = 1 - \frac{1}{18}, \quad x_2 = 1 + \frac{1}{36},$$

so $x_2 > x_1$, while $x_3 = 1 - \frac{1}{54} < x_2$. Hence (x_n) is not monotone.

Quasi-monotone: compute the one-step difference

$$\begin{aligned} x_{n+1} - x_n &= \frac{(-1)^{n+1}}{18(n+1)} - \frac{(-1)^n}{18n} \\ &= -(-1)^n \frac{1}{18}\Big(\frac{1}{n} + \frac{1}{n+1}\Big) \\ &\geq -\frac{1}{18}\Big(\frac{1}{n} + \frac{1}{n+1}\Big) \\ &\geq -\frac{1}{9n}. \end{aligned}$$

For every $n \geq 1$ we have $\dfrac{1}{9n} \leq \dfrac{1}{6}$. Therefore

$$x_{n+1} - x_n \geq -\frac{1}{6} \quad \text{for every } n,$$

so (x_n) is quasi-monotone increasing.

(ii) The sequence y_n.

Not monotone: since $y_{2k} = 0$ but $y_{2k+1} = \dfrac{1}{18}(1 + \frac{1}{2k+1}) > 0$, the sequence goes up and down; e.g. $y_1 > y_2$ and $y_2 < y_3$, so (y_n) is not monotone.

Quasi-monotone: check the two parities.

- If n is even, $x_n = 0$ and $x_{n+1} = \dfrac{1}{18}(1 + \dfrac{1}{n+1}) \geq 0$, hence

$$x_{n+1} - x_n \geq 0 \geq -\frac{1}{6}.$$

- If n is odd, $x_n = \dfrac{1}{18}(1 + \dfrac{1}{n})$ and $x_{n+1} = 0$, so

$$x_{n+1} - x_n = -\frac{1}{18}\left(1 + \frac{1}{n}\right).$$

Now

$$\frac{1}{18}\left(1 + \frac{1}{n}\right) \leq \frac{1}{18} \cdot 2 = \frac{1}{9} < \frac{1}{6},$$

so $x_{n+1} - x_n \geq -\dfrac{1}{6}$ for every odd n as well.

Thus $y_{n+1} \geq y_n - \dfrac{1}{6}$ for every n, so (y_n) is quasi-monotone.

(c) *Does every bounded quasi-monotone increasing sequence converge?*

No. The sequence (y_n) above is a counterexample.

Reason: (y_n) is bounded (all terms lie in $[0, \frac{1}{9}]$) and we just checked it is quasi-monotone increasing, yet it does not converge because the even subsequence is constant 0 while the odd subsequence tends to 1/18 (indeed $y_{2k} = 0$ for all k, and $y_{2k-1} = \frac{1}{18}(1 + \frac{1}{2k-1}) \to \frac{1}{18}$). Two distinct subsequential limits show (y_n) does not converge.

(d) *Comparison with the Monotone Convergence Theorem.*

The Monotone Convergence Theorem states: every upper bounded monotone increasing sequence converges (same for decreasing and lower bounded). The essential feature used in the proof is that monotonicity forbids any downward step: once the sequence rises it can never return below its earlier values, so the tail is trapped and the supremum/infimum arguments force the limit.

By contrast, quasi-monotonicity (as defined here) permits *small* downward steps of size up to $1/6$ at infinitely many indices. Even though each permitted drop is uniformly small, the sequence can still oscillate indefinitely between two different levels (as the example (y_n) shows: it repeatedly attains 0 and values near $1/18$). Therefore boundedness alone does not prevent oscillation under the weakened quasi-monotone condition, and convergence can fail.

Illustration with (y_n)*:* the allowed downward error $1/6$ is large enough to permit the one-step drop from $y_{2k-1} \approx 1/18$ down to $y_{2k} = 0$, so the sequence keeps alternating between two levels. Monotonicity would forbid such drops and hence force convergence; quasi-monotonicity does not.