International Institute of Information Technology, Hyderabad
MA4.101-Real Analysis (Monsoon-2025)

Mid-Semester Exam

Time: 90 Minutes | Total Marks: 40

General Guidelines

- Attempt **any four questions only**.
- One **A4 cheat sheet** (both sides) is allowed.
- Unless stated otherwise, standard theorems from class may be used.
- Show **all steps clearly**; unsupported answers may not receive full credit.

Question (1) [10 Marks] Answer the following.

(a) [**3 Marks**] Prove that for all natural numbers $n \geq 0$,

$$(3+\sqrt{5})^n + (3-\sqrt{5})^n$$

is an even number.

(b) [**3 Marks**] Consider $c \in \mathbb{R}$. Let $S_1 \subset \mathbb{R}, S_2 \subset \mathbb{R}$ be two neighbourhoods of c. Then prove that $S_1 \cup S_2$ is a neighbourhood of c.

(c) [**4 Marks**] Let $S \subset \mathbb{R}$. Then S is an open set iff $S = \text{int}(S)$, where $\text{int}(S)$ is the set of all interior points of S.

Question (2) [10 Marks] Answer the following.

(a) [**3 Marks**] Let $N \geq 1$ be an integer. Show that there exists an integer m with

$$\frac{m}{N} \leq \sqrt{2} < \frac{m+1}{N}.$$

Deduce that $0 \le \sqrt{2} - \frac{m}{N} < \frac{1}{N}$.

(b) **[3 Marks]** Conclude that $\sqrt{2}$ can be approximated arbitrarily well by rationals of the form m/N with $m \in \mathbb{Z}$, and hence that $\mathbb{Q}$ is dense in $\mathbb{R}$.

(c) **[4 Marks]** For $N = 50$, produce an explicit rational with denominator 50 lying below $\sqrt{2}$ and verify the error bound $< 1/50$ using inequalities.

Question (3) [10 Marks] Answer the following.

(a) **[3 +3 Marks]** Consider two sequences (s_n) and (t_n) such that $\lim_{n\to\infty} s_n = s$ and $\lim_{n\to\infty} t_n = t$. Then prove that

(i) $\lim_{n\to\infty} (s_n + t_n) = s + t$.

(ii) $\lim_{n\to\infty} (s_n t_n) = st$.

(b) **[4 Marks]** Let (u_n), (v_n), (w_n) be three sequences of real numbers and $\forall n \in \mathbb{N}$, suppose

$$u_n < v_n < w_n.$$

If $\lim_{n\to\infty} u_n = \lim_{n\to\infty} w_n = l$, then prove that $\lim_{n\to\infty} v_n = l$.

Question (4) [10 Marks] Let $(x_n)_{n\ge1}$ be a sequence and define

$$y_n = \frac{x_n}{x_{n+1}} \qquad (n \ge 1). \tag{1}$$

Answer the following.

(a) **[2 Marks]** Suppose the sequence (x_n) is convergent and $\lim_{n\to\infty} x_n = L$ with $L \in \mathbb{R}$. Can one conclude that $\lim_{n\to\infty} y_n = 1$? Prove it if yes, or give the correct necessary condition(s) if not.

(b) **[3 Marks]** Give an example of $(x_n)_{n\ge1}$ such that $\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} y_n = -1$.

(c) **[3 Marks]** Give another example of $(x_n)_{n\ge1}$ such that $\lim_{n\to\infty} x_n = 0$ but $(y_n)_{n\ge1}$ does not converge to any finite value.

(d) [**2 Marks**] Suppose instead that (x_n) is monotonically increasing and $x_n \to L \neq 0$. Do your conclusions change?

Question (5) [10 Marks] Consider the sequence $(a_n)_{n\geq 1}$ with $a_n = \frac{(-1)^n}{2}$. Answer the following questions.

(a) [**1 Mark**] Does the sequence $(a_n)_{n\geq 1}$ converge?

(b) [**5 Mark**] Let $L \geq 1/2$ be a positive rational number. Define

$$d_n = \inf_{k\geq n} |a_k - L|.$$

Does the sequence $(d_n)_{n\geq 1}$ converge? If no, why? If yes, what is $\lim_{n\to\infty} d_n$? What are the limit points of the sequence $(d_n)_{n\geq 1}$?

(c) [**4 Mark**] Define

$$e_n = \min_{1\leq k\leq n} |a_k - L|.$$

Does the sequence $(e_n)_{n\geq 1}$ converge? If no, why? If yes, what is $\lim_{n\to\infty} e_n$?

Question (6) [10 Marks] Define the sequence $(a_n)_{n\geq 1}$ by $a_1 = 4$ and

$$a_n = \begin{cases} -1 + \frac{1}{k}, & \text{if } n = k! \text{ for some integer } k \geq 2, \\ 1 - \frac{1}{n}, & \text{otherwise.} \end{cases}$$

Here $k! = k \times (k-1) \times \cdots \times 2 \times 1$ for $k \geq 2$. Answer the following questions.

(a) [**1 Mark**] Write out the first six terms $a_1, \ldots, a_6$.

(b) [**2 Marks**] Show that the sequence $(a_n)_{n\geq 1}$ is both bounded from below and bounded from above.

(c) [**3 Marks**] Compute $\sup(a_n)_{n\geq 1}$ and $\inf(a_n)_{n\geq 1}$.

(d) [**4 Marks**] Compute $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$.

Question (7) [10 Marks] A sequence $(x_n)_{n\geq 1}$ of real numbers is said to be *monotone increasing* if $x_{n+1} \geq x_n$ for all $n \geq 1$. We say $(x_n)_{n\geq 1}$ is *quasi-monotone increasing* if $x_{n+1} \geq x_n - \frac{1}{6}$ for all $n \geq 1$. Answer the following.

(a) **[2 Marks]** Prove that every monotone increasing sequence is quasi-monotone increasing as well.

(b) **[4 Marks]** Let $x_n = 1 + \dfrac{(-1)^n}{18n}$ and

$$y_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{18}\left(1+\frac{1}{n}\right) & \text{if } n \text{ is odd} \end{cases}$$

Prove that $(x_n)_{n\geq 1}$ and $(y_n)_{n\geq 1}$ are quasi-monotone increasing sequences that are not monotone increasing.

(c) **[2 Marks]** Prove or disprove: every bounded quasi-monotone increasing sequence converges.

(d) **[2 Marks]** Compare this with the Monotone Convergence Theorem. What essential feature of true monotonicity is lost in the quasi-monotone case, and why does this allow bounded quasi-monotone sequences to fail to converge?

Question (8) [10 Marks] Prove that in $\mathbb{R}$

(a) **[5 Marks]** Every Cauchy sequence is convergent.

(b) **[5 Marks]** Every convergent sequence is Cauchy.