Theorem (existance of least upper Bound)
Let E be a non-empty subset of $\mathbb{R}$ and has an upper bound M.
Then E must have exactly one least upper bound

Proof: We only need to show that E has at least one upper bound

Fix $n \geq 1$. Then for fixed M, n, $\exists$ integer K s.t

$$\frac{K}{n} \geq M \quad (\text{Archemedian Property})$$

$\frac{K}{n}$ is an upper bound on E

Since E is non-empty, let $x_0 \in E$ for fixed n and x_0. $\exists$ integer L s.t

$$\frac{L}{n} < x_0$$

$\frac{L}{n}$ is not an upper bound on E

Further

$$\frac{L}{n} < x_0 \leq M \leq \frac{K}{n} \Rightarrow L < K$$

Let us define a set

$$A := \{ m \in \mathbb{Z} : L < m \leq K \ \& \ \frac{m}{n} \text{ is an upper bound on } E \}$$

$\frac{K}{n} \in A \Rightarrow A$ is non-empty

Define $m_n = \min A$

By definition $\frac{m_n}{n}$ is an upper bound on E

since this is min any element less than this will not belong to A. $\frac{m_n}{n}$ is upper bound and $\frac{m_n - 1}{n}$ is not an upper bound

$$\begin{cases} \dfrac{m_n - 1}{n} < \dfrac{m_{n'}}{n'} \\ \dfrac{m_{n'}-1}{n'} < \dfrac{m_n}{n} \end{cases} \forall\, n, n' \geq 1$$

$\frac{m_1}{1}, \frac{m_2}{2}, \frac{m_3}{3}$ ------ upper bounds

$\frac{m_1-1}{1}, \frac{m_2-1}{2}, \frac{m_3-1}{3}$ --- Not upper bounds

Claim: $\left(\frac{m_n}{n}\right)_n$ is a Cauchy Sequence

$$\left|\frac{m_n}{n} - \frac{m_{n'}}{n'}\right| < \epsilon \quad \forall\, \epsilon > 0 \;\&\; n, n' \geq N$$

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} < \frac{1}{n}$$

$$\frac{m_n}{n} - \frac{m_{n'}}{n'} < -\frac{1}{n'},$$

For $n, n' \geq N$ we have

$$-\frac{1}{N} < \frac{m_n}{n} - \frac{m_{n'}}{n'} < \frac{1}{N}$$

$$\left|\frac{m_n}{n} - \frac{m_{n'}}{n'}\right| < \frac{1}{N} < \epsilon$$

{ For any ϵ one can always choose some N s.t $\frac{1}{N} \leq \epsilon$

$$\lim_{n\to\infty}\left(\frac{m_n}{n}\right) = S \quad (\text{real number})$$

$$\lim_{n\to\infty}\left(\frac{1}{n}\right) = 0$$

So

$$S = \lim_{n\to\infty}\left(\frac{m_n}{n}\right) = \lim_{n\to\infty}\left(\frac{m_n - 1}{n}\right)$$

a) Claim S is an upper bound

By def. $\frac{m_n}{n}$ is an upper bound on E

$$x \leq \frac{m_n}{n}$$

$$\lim_{n\to\infty}(x) \leq \lim_{n\to\infty}\left(\frac{m_n}{n}\right)$$

b) Assume y is another upper bound on E

by def. $\frac{m_n - 1}{n}$ is not an upper bound on E

$$\frac{m_n - 1}{n} < y$$

$$\lim_{n\to\infty}\left(\frac{m_n - 1}{n}\right) \leq \lim_{n\to\infty}(y) \Rightarrow S \leq y$$

you can equality by taking Limit

Hence prooved

Least upper bound is also called Supremum

Sup(E) = least upper bound on E

Proposition: ∃ exist a positive real number x such that $x^2 = 2$

take set

$$A = \{ y \in \mathbb{R} : y \geq 0 \ \& \ y^2 < 2 \}$$

take number (any valid)

2 is an upper bound $\forall z \in A \ z \leq 2$ (Proove formally!)

A is a non-empty subset of $\mathbb{R}$ with an upper bound 2 by Prev. theorem A must have a least upper bound

$$x := Sup(A)$$

Claim: $x^2 = 2$

Suppose

Case 1 $x^2 < 2$

~~take~~ some ϵ $0 < \epsilon < 1$ s.t

$(x+\epsilon)^2 = x^2 + \epsilon^2 + 2\epsilon x$ $\epsilon^2 < x$ $x \leq 2$

$\leq x^2 + 5\epsilon$

But $x^2 < 2$, then we can find positive ϵ

s.t $x^2 + 5\epsilon < 2$

$x^2 < 2 \Rightarrow (x+\epsilon)^2 < 2$ for some small $\epsilon > 0$

$(x+\epsilon) \in A$ [contradiction as $Sup(A)$]

<u>no element in A</u>

<u>should be larger than $Sup(A)$</u>

Case 2 Suppose $x^2 > 2$ $0 < \epsilon < 1$

again

$(x-\epsilon)^2 = x^2 + \epsilon^2 - 2\epsilon x$ $-x \geq -2$

$x^2 + \epsilon - 4\epsilon$

$x^2 - 3\epsilon$

we ~~ch~~ can choose small ϵ s.t

$x^2 - 3\epsilon > 2$

$(x-\epsilon)^2 > 2$ $(x-\epsilon)^2 < (x)^2$

so this is upper bound

If $(x-\epsilon) \geq y \ \forall \ y \in A$

$(x-\epsilon)$ is an upper bound on $A \Rightarrow$ Contradiction as x is $\text{Sup}(A)$

$(x-\epsilon) < y$

$(x-\epsilon)^2 < y^2 < 2$

A contradiction again

$(x-\epsilon)^2$ cannot be greater than 2

This property of $\text{Sup}(A)$ is unique to real numbers only.

Def $x \geq 0$

$x^{\frac{1}{n}} := \text{Sup}(y \in \mathbb{R} : y \geq 0 \ \& \ y^n \leq x \}$

Ex: Proove that $\{y \in \mathbb{R} : y \geq 0 \ \& \ y^n \leq x\}$ has an upper bound

Convergence of Sequences

Lets go gigs RA

ϵ - close to L

Let $(a_n)_{n=m}^{\infty}$ be a sequence of Real numbers

we say that $(a_n)_{n=m}^{\infty}$ is ϵ - close to some real number L if for $\epsilon > 0$

$$|a_n - L| \leq \epsilon \quad \forall n \geq m$$

seq starts at m

eventually ϵ - close to L if $\exists N \geq m$ s.t

$$|a_n - L| \leq \epsilon \quad \forall n \geq N$$

$(a_n)_{n=m}^{\infty}$ is said to be convergent if $\exists$ ~~$|a_n - L|$~~ $N \geq m$ s.t $|a_n - L| \leq \epsilon \ \forall \epsilon > 0$ and $n \geq N$ and $L \in \mathbb{R}$

Claim : A sequence $(a_n)_{n=m}^{\infty}$ of reals converge to two points L and $L' \in \mathbb{R}$

Assume

$\exists N$ s.t $|a_n - L| \leq \epsilon \ \forall \epsilon > 0$ and $n \geq N$

$\exists M$ s.t $|a_n - L'| \leq \epsilon \ \forall \epsilon > 0$ and $n \geq M$

choose $|L - L'|/3$

$|L-L'| = |L - a_n + a_n - L'|$

$|L-L'| \leq |L-a_n| + |a_n - L'|$

$3\epsilon \leq 2\epsilon$

Contradiction

$L = L'$

Limit of a sequence $(a_n)_{n=m}^{\infty}$ then limit of the sequence is defined as

$$L := \lim_{n\to\infty} (a_n)$$

Relation between

$\underset{n\to\infty}{\mathrm{LIM}}\, a_n$ and $\lim_{n\to\infty} a_n$

Claim: Let $(a_n)_{n=m}^{\infty}$ be a cauchy sequence of Rationals

then $\underset{n\to\infty}{\mathrm{LIM}}\, a_n = \lim_{n\to\infty} a_n$

will do next class.

Lemma: Eevery convergent series of reals is a Cauchy seq.

let $(a_n)_{n=m}^{\infty}$ be a convergent seq.

$\exists N \geq M \quad |a_n - L| < \epsilon \;\forall \epsilon > 0$ and $n \geq N$

$-\epsilon < a_n - L < \epsilon$

$$|a_n - a_m| = |a_n - L + L - a_m|$$
$$\leq |a_n - L| + |a_m - L|$$
$$\leq \epsilon \;\; \forall \; n, m \geq N \text{ and } \forall \epsilon > 0$$

(a_n) is a Cauchy seq.

Cause $|a_n - L| < \epsilon/2$ for some N'
$|a_m - L| < \epsilon/2$ " M