Lemma: Let $x \neq 0$ be a real number, then $x = \underset{n\to\infty}{LIM}\, a_n$ for some Cauchy sequence $(a_n)_{n\geq 1}$ that is bounded away from zero.

Lemma: If $(a_n)_{n\geq 1}$ is a Cauchy sequence bounded away from zero, then $(a_n^{-1})_{n\geq 1}$ is a Cauchy sequence.

For all $x \neq 0$, we can define

$$x^{-1} = \underset{n\to\infty}{LIM}\, a_n^{-1} \quad \text{where } (a_n) \text{ and } (b_n) \text{ are equivalent.}$$

where $x = \underset{n\to\infty}{LIM}\, a_n = \underset{n\to\infty}{LIM}\, b_n$

$(\dots a_1, \dots\dots a_n, \dots) = (a_n)$

$|a_i| \geq \delta$

$|a_i - a_j| \leq \varepsilon \quad \forall\, i,j \geq N$

$(b_n) = (\delta \dots \delta, a_1 \dots a_n, \dots)$

(a_n) & (b_n) are equivalent.

Claim:

If $x = \underset{n\to\infty}{\text{LIM}}\, a_n$, $y = \underset{n\to\infty}{\text{LIM}}\, b_n$ for some cauchy sequences (a_n) & (b_n) that are bounded away from zero.

If $x = y$

Then $x^{-1} = y^{-1}$

or $\underset{n\to\infty}{\text{LIM}}\, a_n^{-1} = \underset{n\to\infty}{\text{LIM}}\, b_n^{-1}$

Proof:

$$A = \left(\underset{n\to\infty}{\text{LIM}}\, a_n^{-1}\right)\left(\underset{n\to\infty}{\text{LIM}}\, a_n\right)\left(\underset{n\to\infty}{\text{LIM}}\, b_n^{-1}\right) = \underset{n\to\infty}{\text{LIM}}\,(a_n^{-1} \times a_n \times b_n^{-1}) = y^{-1}$$

$$= \left(\underset{n\to\infty}{\text{LIM}}\, a_n^{-1}\right)\left(\underset{n\to\infty}{\text{LIM}}\, b_n\right)\left(\underset{n\to\infty}{\text{LIM}}\, b_n^{-1}\right) = x^{-1}$$

Hence, proved.

Def:

Positive / Negative real numbers

$x = \underset{n\to\infty}{\text{LIM}}\, a_n$ is a positive / negative real number iff $(a_n)_{n\geq 1}$ is a positively / negatively bounded away from zero Cauchy sequence.

$x > y$ iff $x - y$ is a positive real number.

Lemma: Let $(a_n)_{n\geq 1}$ be a Cauchy sequence of non-negative rationals. Then,

$$\underset{n\to\infty}{\mathrm{LIM}}\, a_n \geq 0$$

Proof: Let us call $x = \underset{n\to\infty}{\mathrm{LIM}}\, a_n$

Assume for contradiction that x is negative.

Then $x = \underset{n\to\infty}{\mathrm{LIM}}\,(b_n)$ for some Cauchy sequence $(b_n)_{n\geq 1}$ that is negatively bounded away from zero.

$\Rightarrow$ $b_n \leq -c$ for some $c > 0$ $\forall n$

We know that $a_n \geq 0$

Then $a_n - b_n \geq c$
$> c/2$

or $|a_n - b_n| > c/2$ $\forall n$

If $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ are equivalent then there exists $N \geq 1$ s.t.

$$|a_n - b_n| \leq \varepsilon \quad \forall\ i, j \geq N \quad \forall\ \varepsilon > 0$$

For $\varepsilon = c/2$ we have a contradiction.

- If $a_n \geq b_n$ then $\underset{n\to\infty}{LIM}\, a_n \geq \underset{n\to\infty}{LIM}\, b_n$, (a_n) and (b_n) are Cauchy sequence.

  $a_n - b_n \geq 0 \Rightarrow (a_n - b_n)_n$ is a Cauchy sequence of non-negative rationals

  From Lemma,

  $$\underset{n\to\infty}{LIM}\,(a_n - b_n) \geq 0$$

  $$\Rightarrow \underset{n\to\infty}{LIM}\, a_n \geq \underset{n\to\infty}{LIM}\, b_n$$

- **Proposition**: Let x be a positive real number. Then there exists a rational $q > 0$ and an integer $N > 0$ s.t.

  $$q \leq x \leq N$$

  Proof: Since $x > 0$, we have

  $x = \underset{n\to\infty}{LIM}\, a_n$ for some Cauchy sequence $(a_n)_{n\geq}$ that is positively bounded away from zero.

  $\Rightarrow$ ① $a_n \geq c$ for some $c > 0$ $\forall n$ rational

  ② $a_n = |a_n| \leq N$ for some $N > 0$.

  ① & ② $c \leq a_n \leq N$

$C_n = (c)_{n\geq 1}$
$N_n = (N)_{n\geq 1}$

$$c = \lim_{n\to\infty} C_n \leq \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} N_n = N$$

$$\underbrace{C \leq x \leq N_n}_{\text{Archimedian property}}$$

e.g. Let $x>0$ and $y>0$ then $\frac{x}{y}>0$

$$\frac{x}{y} \leq N < N+1 = M \Rightarrow \frac{x}{y} < M \text{ for some +ve integer.}$$

$x=1$ then $\frac{1}{y} < M$ for some $M \Leftrightarrow \frac{1}{M} < y$ for some M.

- **Lemma**: Let x, y be two real numbers with $x<y$. Then there exists a rational q s.t.

$$x < q < y$$

Proof: Since $y-x>0$

From Archimedian property there exists $n>0$ s.t. $y-x > \frac{1}{n}$

Now let $A = \{m \in \mathbb{Z} : x < \frac{m}{n}\}$ for fixed n

A is non-empty from Archimedian property.

define $m_0 = \min A$

$\Rightarrow$ $x < \frac{m_0}{n}$ because $m_0 \in A$

Now for $\underset{\notin A}{\frac{m_0-1}{n}}$; $x \geq \frac{m_0-1}{n}$

We have,

$$\frac{m_0-1}{n} \leq x < \frac{m_0}{n}$$

Now if $y > \frac{m_0}{n}$, then this concludes the proof

$$x < \underbrace{\frac{m_0}{n}}_{q} < y$$

if $y \leq \frac{m_0}{n}$, then

$$y - x \leq \frac{m_0}{n} - \frac{m_0-1}{n}$$

$$= \frac{1}{n}$$

$\Rightarrow$ $y - x \leq \frac{1}{n}$ which is a contradiction

Hence, y can only be $> \frac{m}{n}$.

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For reals; we defined

Addition, subtraction, inverse, multiplication, absolute value

$\Rightarrow$ All the laws of algebra that are true for rationals are also true for reals.

- **Upper bound on subset of $\mathbb{R}$**

Let $E \subseteq \mathbb{R}$. Then E is said to have an upper bound $M \in \mathbb{R}$ iff $x \leq M$ for all $x \in E$

$\Updownarrow x \leq M'$

Least upper bound on E:-

$M \in \mathbb{R}$ is a least upper bound on E iff

i) M is an upper bound

ii) Any other upper bound M' on E satisfies $M' \geq M$

- **Claim:** A subset E of $\mathbb{R}$ can have at most ~~two~~ one least upper bound.

Proof: Assume M_1 & $M_2 \in \mathbb{R}$ are two least upper bounds on E.

① Since M_1 is a least upperbound, $M_2 \leq M_1$

② also, $M_1 \geq M_2$

So, $M_1 = M_2$

$-\infty \rightarrow$ least upper bound of set $\emptyset$

$+\infty \rightarrow$ least upper bound of non-bounded set

- **Theorem:** Any non-empty subset E of $\mathbb{R}$ has exactly one least upper bound if it has an upper bound.

Proof: We need to show that there is at least one least upper bound on E.

Let M be an upper bound on E (Given in premise)

Then for any $n \geq 1$, we can find integer K s.t. $K/n \geq M$
(Archimedian property as for fixed M, n $\frac{M}{n} \leq$ some integer)

$\Rightarrow$ $\frac{K}{n}$ is an upper bound on E $\forall$ $n \geq 1$

Let $x_0 \in E$ (because E is non empty)

then we can find

$L/n < x_0$ (Again from Archimedian property)

$\Rightarrow$ L/n is not an upper bound on E

$$\frac{L}{n} < x_0 \leq \frac{K}{n} \Rightarrow L < K$$