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Definitions:

① Sequence: A sequence $(a_n)_{n=m}$ for some $m \in \mathbb{Z}$ is any mapping from $\{n : n \geq m ; m \in \mathbb{Z}\}$ to $\mathbb{Q}$ as a collection of rationals $(a_m, a_{m+1}, \cdots)$

② ε-steadiness: For $\varepsilon \in \mathbb{Q}$ and $\varepsilon > 0$, a sequence $(a_n)_{n=m}$ rational is called ε-steady iff $|a_i - a_j| \leq \varepsilon \ \forall i, j \geq 0$.

(b). Eventually ε-steady
iff $\exists N \in \mathbb{N}$ s.t.
$|a_i - a_j| \leq \varepsilon \ \forall i, j \geq N$.

(c) Cauchy sequence
iff $\exists N \in \mathbb{N}$ s.t. for all rationals $\varepsilon > 0$
$|a_i - a_j| \leq \varepsilon \ \forall \ i, j \geq N$.

Ex. $\left(\frac{1}{n}\right)_{n \geq 1}$ is it 1-steady? Yes.
is it 0.1-steady? No.
is it eventually 0.1-steady? Yes for $N = 10$

is it eventually ε-steady for all $\varepsilon > 0$?
Yes.

Proof: $|a_i - a_j| = \left|\frac{1}{i} - \frac{1}{j}\right| \leq \varepsilon \quad \forall \ i, j \geq N \ \forall \varepsilon > 0$

$0 \leq \frac{1}{i}, \frac{1}{j} \leq \frac{1}{N}$

$\left|\frac{1}{i} - \frac{1}{j}\right| \leq \frac{1}{N}$ so if we choose $\frac{1}{N(\varepsilon)} \leq \varepsilon$
$\Updownarrow$
$1/\varepsilon \leq N(\varepsilon)$

We choose such $N(\varepsilon)$. Then $|1/i - 1/j| \leq \varepsilon \ \forall i, j \geq N(\varepsilon)$
$\forall \ \varepsilon > 0$

~~finite~~

$\downarrow$

* **Bounded sequences:** A sequences $(a_n)_{n\geq 0}$ is said to be bounded by M iff $|a_n| \leq M$ $\forall n \geq 0$.

↳ Rational

A sequence is bounded iff it is bounded by some M.

Q. Is a finite sequence always bounded?

Ans Yes.

* $(a_n)_{n=1}^{m} = (a_1, \dots, a_m)$

Is every cauchy sequence bounded?

Q let $(a_1 \dots a_m, a_{m+1}, a_{m+2}, \dots)$ be a cauchy-sequence, i.e $\forall \varepsilon > 0 \; \exists N \in \mathbb{N}$ s.t. $|a_i - a_j| \leq \varepsilon$ $\forall i,j \geq N$.

Ex.

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Def: ε-close sequence: $\varepsilon > 0$
$(a_n)_{n\geq 0}$, $(b_n)_{n\geq 0}$ are said to be ε-close iff
$$|a_n - b_n| \leq \varepsilon \quad \forall \; n \geq 0.$$

— Eventually ε-close sequence
iff $\exists \; N \in \mathbb{N}$ s.t.
$$|a_n - b_n| \leq \varepsilon \quad \forall \; n \geq N$$

$\Rightarrow$ Equivalent sequences
$$|a_n - b_n| \leq \varepsilon \quad \forall \; n \geq N \quad \text{all } \varepsilon > 0$$

Ex. $a_n = 1 + 10^{-n}$ $\quad (a_n)_{n\geq 1}$
$b_n = 1 - 10^{-n}$ $\quad (b_n)_{n\geq 1}$

Are they equivalent. need to claim. ~~there will always exist.~~
Yes,

Proof: $|a_n - b_n| = 2 \times 10^{-n} \leq 2/n \leq \varepsilon \quad \forall \; n \geq N$
claim: $10^n \geq n \quad \forall \; n \in \mathbb{N}$ $\rightarrow$ Prove by induction
$$10^{-n} \leq \frac{1}{n}$$

so we can always choose N s.t.
$$\frac{2}{\varepsilon} \leq N$$

Real number: A real number is an object of the form
$\operatorname{LIM}_{n\to\infty} a_n$ for some cauchy sequence $(a_n)_{n\geq 0}$
of rational numbers. We will say that $\operatorname{LIM}_{n\to\infty} a_n$
$= \operatorname{LIM}_{n\to\infty} b_n$ iff the cauchy sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ are equivalent.

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Addition of reals $x = \lim_{n\to\infty} a_n$ $(a_n), (b_n)$ are cauchy sequence

$y = \lim_{n\to\infty} b_n$

$x+y = \lim_{n\to\infty} (a_n + b_n)$

Is $(a_n+b_n)_{n\geq 0}$ a cauchy sequence?

$|(a_i+b_i) - (a_j+b_j)| \leq \varepsilon \quad \forall i,j \geq N \quad \forall \varepsilon>0$

$\leq$

$|(a_i-a_j)| + |(b_i-b_j)| \leq \varepsilon/2 + \varepsilon/2$

$\forall i,j \geq \max(N_1,N_2) \quad \forall \varepsilon>0$

Multiplication

$|a_i - a_j| \leq \varepsilon/2 \quad \forall i,j \geq N_1 \quad \forall \varepsilon>0$

$x\cdot y = \lim_{n\to\infty} (a_n \cdot b_n)$

$|b_i - b_j| \leq \varepsilon/2 \quad i,j \geq N_2 \quad \forall \varepsilon>0$

$(a_n \cdot b_n)_{n\geq 0}$ is a cauchy sequence for cauchy sequence $(a_n)_{n\geq 0}$ as $(b_n)_{n\geq 0}$

$|a_ib_i - a_jb_j - a_jb_i + a_jb_i| \leq |(a_i-a_j)b_i| + |a_j(b_j-b_i)|$

$= |a_i-a_j||b_i| + |b_j-b_i||a_j|$

$(a_n)_{n\geq 0} = (q)_{n\geq 0} \quad q \in \mathbb{Q}$

$\lim_{n\to\infty} a_n = q \Rightarrow$ Rationals are also reals.

Negation of such $x = \lim_{n\to\infty} a_n$

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$$-x = -\lim_{n\to\infty} a_n$$

$$= \lim_{n\to\infty} (-1) \text{ LIM}_{n\to\infty} a_n = \text{LIM}_{n\to\infty} (-a_n)$$

Reciprocals of reals

$x = \text{LIM}_{n\to\infty} a_n$ $(a_n)_{n\geq 0}$ is a cauchy sequence.

$$x^{-1} = \text{'LIM}_{n\to\infty} (a_n^{-1})$$

Problem e.g. $(10^0, 10^{-1}, 10^{-2}, \dots)$ ... cauchy seq^n.
$(10^0, 10^1, 10^2, \dots)$ ... not a cauchy.

Def^n: Sequence bounded away from zero.
A sequence $(a_n)_{n\geq 0}$ is said to be bounded away from zero iff $\exists$ some rational $c \neq 0$ s.t.
$|a_n| \geq c \quad \forall\, n \geq 0$

(a) Positively bounded away from zero
iff $a_n \geq c \quad \forall\, n \geq 0$ some $c > 0$

(b) Negatively bounded away from zero iff $a_n \leq -c \quad \forall\, n \geq 0$ some $-c < 0$

Positive / Negative reals

$x = \text{LIM}_{n\to\infty} (a_n)$ iff $(a_n)_{n\geq 0}$ is a Positively/ Negatively bounded away cauchy sequence.

$$0 = \lim_{n\to\infty} (0)$$

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Claim: Let $x \neq 0$ be a real number then there exists a bounded away from zero Cauchy sequence $(a_n)_{n\geq 0}$ s.t.

$$\underset{n \to \infty}{\mathrm{LIM}} (a_n) = x.$$

Claim: Let $(a_n)_{n\geq 0}$ be a Cauchy sequence that is bounded away from zero. Then $(a_n^{-1})_{n\geq 0}$ is a Cauchy sequence.