* Reciprocal of a Rational number
  If $x/\!/y$'s reciprocal is defined as $y/\!/x$ $(x \neq 0, y \neq 0)$
- If x is an integer then what is $x/\!/_1$?

$$g: Q \to Z$$
$$g(x/\!/_1) = x$$
$$0/\!/_1 = 0$$

- When a rational number $x/\!/y$ is nonzero
  If $x/\!/y = 0$ then $x = 0,\ y \neq 0$
  If $y/\!/x \neq 0$ then $x \neq 0,\ y \neq 0$

Let $x \in R$
$y \in Q$ be a non zero rational number then

$$x/y = x \cdot \underbrace{y^{-1}}_{\text{reciprocal of } y}$$

Then for $x = x/\!/_1$ $\quad x, y \in Z$
$y = y/\!/_1$ $\quad y \neq 0$ $\qquad N \subseteq Z \subseteq Q$

$$x/\!/y = x/\!/_1 \times 1/\!/y$$
$$= x/\!/y$$

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$\Rightarrow$ Let $a, b, c \in Z$ ST $b \neq 0,\ c \neq 0$ then $ac/_{bc} = a/b$

$$(a/b)(c/c) = ac/bc$$
$$c \times c^{-1} = 1$$

* Law of Algebra for Rationals :-
  . Let $x, y, z \in Q$

i) $x + y = y + x$

ii) $x + (y + z) = (x + y) + z$

iii) $x + (-x) = (-x) + x = 0$

iv) $x + 0 = 0 + x = x$

v) $x \cdot y = y \cdot x$

vi) $x(yz) = (xy)z$

vii) $x \cdot 1 = 1 \cdot x = x$

viii) $x(y + z) = xy + xz$

ix) $(x + y)z = xz + yz$

x) $x \neq 0$ $\quad x \cdot x^{-1} = x^{-1} x = 1$

Absolute value ⇒ Let $x \in \mathbb{Q}$ then absolute value of x is denoted as $|x|$ & is defined as

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

ε-closeness of two rationals. Let $x, y \in \mathbb{Q}$. Then y is said to be ε-close to x for rational $\varepsilon > 0$ iff

$$d(x,y) := |x-y| \le \varepsilon$$

↳ distance b/w x & y

Prop. of absolute values

1) For each $x \in \mathbb{Q}$ $|x| \geq 0$ $|x| = 0$ iff $x = 0$

2) $|x| \le y$ ⟺ $-y \le x \le y$

for any $x \in \mathbb{Q}$

$-x \le |x| \le x$ $\forall$ $x \in \mathbb{Q}$

~~Case 1: If $x \geq 0$ then $|x| = x$~~ $\left\{ \begin{array}{l} x \le x \\ |x| \le x \end{array} \right\}$

~~$-x$ is $\le$ $|x| \geq 0$ +ve no~~

~~Case 2: If $x < 0$ then $|x| = -x$~~

Case 1 If $x \geq 0$ then

$|x| = x$

First $-|x| \le 0 \le x$

second part $x \le x \Rightarrow x \le |x|$

Case 2 $x < 0 \Rightarrow |x| = -x$

$x = -|x| \le |x|$

$x \le x \Rightarrow -(-x) \le x$

$\Rightarrow -|x| \le x$

⇐ Let us assume $-y \le x \le y$

Case 1 ⇒ If $x \geq 0$ then $|x| = x \le y$

Case 2 ⇒ If $x < 0$ then $|x| = -x \le y$

⇒ Assume $|x| \le y$

case 1 $x \geq 0$ then from assumption

$|x| = x \le y \Rightarrow x \le y$

Also $y \geq |x| \geq 0$

$\Rightarrow -y \le 0$

$x \geq -y$

$|x| \geq -y$

Case 2 : $x < 0$

iii) $|x+y| \le |x| + |y|$

Proof $\Rightarrow$

$$-|x| \le x \le |x|$$
$$-|y| \le y \le |y|$$

Add

$$-|x| - |y| \le x+y \le |x| + |y|$$
$$-(|x|+|y|) \le x+y \le (|x|+|y|)$$

iv) $|y||x| = |xy|$

v) $|x^{-1}| = |x|^{-1}$

vi) $|-x| = |x|$

* $d(x,y) := |x-y|$

1) $d(x,y) \ge 0 \quad \forall\ x, y \in \mathbb{Q}$ & $d(x,y) = 0$ iff $x = y$ (non-negativity prop)

2) $d(x,y) = d(y,x)$

3) $d(x,y) \le d(x,w) + d(w,y)$ for all $w \in \mathbb{Q}$ (Δ'le inequality)

* Prop of ε-closeness

1) If x is ε-close to y the vice-versa is also true

2) If $d(x,y) \le \varepsilon$ $\quad \varepsilon, \delta > 0$

$d(y,z) \le \delta$

then $d(x,z) \le \varepsilon + \delta$

3) Let $d(x,y) \le \varepsilon$ & $d(x,z) \le \varepsilon$ Then for any w b/w y & z then

$$d(x,w) \le \varepsilon$$

$w \ne y$ & $w \ne z$ bcs for $w = z$ or $w = y$ the proof is immediate

i) $y < w < z$ $\quad$ ii) $z < w < y$

Then $w = (1-t)y + tz$ where t is b/w 0, and 1 $\quad t \in \mathbb{Q}$

$$w = y + t(z-y)$$
$$w - y = t(z-y)$$
$$t = w-y/z-y$$

Then $d(x,w) = |x-w|$

$$= |x - (1-t)y - tz|$$
$$= |tx + (1-t)x - (1-t)y - tz|$$

$= |t(x-z) + (1-t)(x-y)|$
$\le |t(x-z)| + |(1-t)(x-y)|$
$= t|x-z| + (1-t)|x-y|$
$= t\, d(x,z) + (1-t)\, d(x,y)$
$\le t\varepsilon + (1-t)\varepsilon = \varepsilon$

* Exponentiation to a natural number

Let $x \in \mathbb{Q}$ then define

$$x^0 := 1 \quad (0^0 = 1)$$

If x^n is defined inductively for $n \in \mathbb{N}$ then

$$x^{n+1} = x^n \cdot x$$

properties → 1) $x^n x^m = x^{n+m} = x^m \cdot x^n$
2) $(x^n)^m = x^{nm} = (x^m)^n$
3) $|x^n| = |x|^n$

Def → $x^{-n} = \frac{1}{x^n}$

Then one can talk abt x^n, where $x \in \mathbb{Q}$, $n \in \mathbb{Z}$

Lemma → Let n be a natural no. and q be +ve natural no. Then $\exists$ unique m & r with $0 \le r < q$ ST

$$n = mq + r \quad \text{(div. algo.)}$$

Proof → we will apply induction on n for fixed $q > 0$

a) Base case for $n = 0$ if we choose $m = 0$, $r = 0 < q$

$$0 = 0q + 0$$

b) assume that $n = mq + r$ with $0 \le r < q$

Then we need to ST $\exists$ m', r' with $0 \le r' < q$ ST

$$n+1 = m'q + r'$$

case 1 If $n+1 < q$ then take

$$\left\{ \begin{array}{l} r' = n+1 \text{ and } m' = 0 \text{ i.e} \\ n+1 = 0 \times q + (n+1) \end{array} \right\}$$

Case 2 If $n+1 \ge q$, $0 < q$

$$\underbrace{n+1-q}_{= m'q + r'} \le n \qquad 0 \le r' < q$$

$$n+1-q = m'q + r'$$

$$n+1 = (m'+1)q + r'$$

$m = m'+1 \qquad r = r'+1$