Cardinality of sets

size of set = # of elements

Def (equal cardinality) → We say that sets X & Y have equal cardinality iff there exists a bijective f^n from X to Y

⇒ To prove N, $B = \{2n : n \in N\}$
$C = \{2n+1 : n \in N\}$
all 3 have same cardinality

$f: N \to B$ ST
$f(x) = 2x \quad \forall \; x \in N$

N & B
N & C have same cardinality

" Equal cardinality as a rel^n is an equivalence rel^n

★ Claim ⇒ Let $f: X \to Y$ & $g: Y \to Z$ be bijective f^n then
$$(g\circ f)^{-1} = f^{-1} \circ g^{-1}$$

Proof ⇒
$g\circ f : X \to Z$ (bijective f^n)

Let $h: Z \to X$ be inverse of $g\circ f$

Then $h\circ(g\circ f)(x) = x$

Let $g\circ f(x) = z \Rightarrow x = (g\circ f)^{-1}(z)$ — ②

$\Rightarrow g^{-1}\circ(g\circ f)x = g^{-1}(z)$

$f(x) = g^{-1}(z)$

$x = (f^{-1}\circ g^{-1})(z)$ — ③

From ② & ③ $(g\circ f)^{-1} z = (f^{-1}\circ g^{-1})(z) \quad \forall \; z \in Z$

$(g\circ f)^{-1} = f^{-1}\circ g^{-1}$

□

★ Def (cardinality) ⇒ Let $n \in N$. Then X is said to have cardinality n iff there exists a bijection from X to $\{i : 1 \le i \le n\}$ i.e

$f: X \to \{i \in N : 1 \le i \le n\}$ is a bijection
$\qquad\qquad \| $
$\{1, 2, \ldots, n\}$ → cardinality

Cardinality =: No. of elements in X

Q) Say X has cardinality $n \geq 1$ Then what is cardinality of $X \setminus \{x\}$ for some $x \in X$?

remove element x from set X

Cardinality is m where $m_{++} = n$

* Proposition ⇒ The cardinality of set X is unique

Def (finite set) ⇒ A set X is said to be finite iff ∃ an $n \in N$ ST cardinality of X is n. Otherwise the set is called "infinite"

Lemma ⇒ N is infinite

Proof ⇒ Suppose X is finite i.e ∃ bijective fxn $f: \{i \in N, 1 \leq i \leq n\} \to X$ (for contradiction)

Claim → ∃ $M \in N$ S.T $f(i) \leq M$ ∀ $1 \leq i \leq M$

$X = \{f(1) \dots f(n)\}$ — (1)

Each $f(i) \leq M$

But M_{++} is also a natural number while it is not there in N from (1)

This a contradiction ⇒ N is not finite □

Proof of claim ⇒ (By induction)

(a) Base case $(n=1)$

we need to show that ∃ $M(1)$ s.t

$f(1) \leq M(1)$

Choose $M(1) = f(1)$

(b) Assume $f(i) \leq M(n)$ ∀ $1 \leq i \leq n$

$M(1) = f(1)$

⋮

$M(n) = \max\{M(l), f(n)\}$ where $l_{++} = n$

$M(n+1) = \max\{M(n), f(n+1)\}$

ST for $1 \leq i \leq n+1$ $f(i) \leq M(n+1)$

We need to show that $f(i) \leq M(n+1)$ ∀ $1 \leq i \leq n+1$

Integers

If $a, b \in N$ then $a-b$ is called an integer

Let $c-d$ be another integer. Then

1. $a-b = c-d$ iff $a+d = b+c$
2. $(a-b) + (c-d) := (a+c) - (b+d)$
3. $(a-b) \times (c-d) = (ac+bd) - (ad+bc)$

Replacement prop. $\begin{cases} a-b = a'-b' \\ \text{then } (a-b)+(c-d) = (a'-b')+(c-d) \end{cases}$

consider $n-0 \ \forall \ n \in N$

We can find a bijective mapping from

$$\{n-0 : n \in N\} \text{ to } N$$

$$f(n-0) = n \quad \forall \ n \in N$$

Define / Identify →

$n-0$ with n, i.e

$$n-0 := n$$

$$n_{++} - 0_{++} = n$$

Def (Negation of integers) ⇒ Let $a-b$ be an integer for $a,b \in N$

Then negation of $a-b$, denoted $-(a-b)$ is defined as $(b-a)$

Subtraction of integer : Let x, y be integers then

$$x-y = x+(-y)$$

Notice : Let $x = x-0$

$y = y-0$

be two integers

$$x-y = (x-0)+(0-y)$$

$$= x+0 - 0+y$$

$$= x-y$$

* Thm

↳ all integers are either 0, N or $(-N)$

Rational Numbers

Let $x, y \in Z$ be two integers such that $y \neq 0$ then the no. $x//y$ are called rational no.

Then $x//y$ will be said equal to $z//t$ when $z, t \in Z$ & $t \neq 0$, i.e

$$x//y = z//t \Rightarrow xt = zy$$

① $x//y + z//t = (xt+zy)//yt$

② $x//y \times z//t = (xz)//(yt)$

③ $-(x//y) = (-x)//y$

④ $x//0$ is not considered as a rational number

* Reciprocal of a Rational number

If $x//y$ is reciprocal is defined as $y//x$ $(x \neq 0, y \neq 0)$

- If x is an integer then what is $x//_1$?

$g : Q \to Z$

$g(x/_1) = x$

$0//_1 = 0$

- When a rational number $x//y$ is nonzero

If $x//y = 0$ then $x = 0$, $y \neq 0$

If $y//x \neq 0$ then $x \neq 0$, $y \neq 0$

Let $x \in R$

$y \in Q$ be a non zero rational number then

$x/y = x \cdot y^{-1}$

↳ reciprocal of y

Then for $x = x//_1$

$y = y//_1$

$x, y \in Z$

$y \neq 0$

$N \subseteq Z \subseteq Q$

$x//y = x//_1 \times 1//y$

$= x//y$

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