Lec 3

•) Functions

Let X & Y be two sets. Let $P(x,y)$ be a property such that for every $x \in X$, there is exactly one $y \in Y$ s.t. $P(x,y)$ is true. Then a function $f: X \to Y$ is defined by P as follows:

→ For each $x \in X$, function f assigns an object $f(x) \in Y$ s.t. $P(x, f(x))$ is true.

X = Domain | $f(x) = x+5$ [Explicit definition of f]

Y = Co-Domain | ~~$y = x+5$~~

① Injective / one to one function:

$f: X \to Y$ is one to one iff

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \quad \forall \; x_1, x_2 \in X$

② Surjective / onto function: [Codomain = Range]

$f: X \to Y$ is surjective iff.

for each $y \in Y$ $\exists$ an $x \in X$ s.t

$y = f(x)$ or $P(x,y)$ is true.

③ Bijective Function:

Injective + surjective.

• Ex: $f: N \to N$

$y = 2x$

① Injective ✓

② Surjective ✗

Can become surjective if $Y = \{2n : n \in N\}$

⇒ Then $f: N \to Y$ is bijective

- When are 2 functions equal?

A) Domain & Range should be same

OR

$f: X \to Y$

$g: X \to Y$

$f = g$ iff $f(x) = g(x) \ \forall \ x \in X$

- Composition of 2 functions:

$f: X \to Y$ $\qquad$ $g: Y \to Z$

$f \circ g$ is defined as:

$(f \circ g)(y) = f(g(y)) \ \forall \ y \in Y$

- $h: T \to L$ then $\qquad h \circ (f \circ g) = (h \circ f) \circ g$ $\qquad$ Associative Prop of composition of func.

- Inverse of a Function:

Let $f: X \to Y$ be a $\boxed{\text{bijective}}$ function then inverse of f is defined as:

$$f^{-1} \circ f(x) = x \quad \forall \ x \in X$$

$$\Updownarrow$$

$$f \circ f^{-1}(y) = y \quad \forall \ y \in Y$$

(i) $f^{-1} \circ f = 1_{X \to X}$

(ii) $f \circ f^{-1} = 1_{Y \to Y}$

] Identity Function.

• Image & Inverse image :

(1) Let $f : X \to Y$ be a function. Let $A \subseteq X$. Then image of A under f is a set defined by

$$f(A) = \{ y \in Y : y = f(x) \ \forall x \in A \}$$

(2) Let $f : X \to Y$ and $B \subseteq Y$. then inverse image of B under f is ~~called~~ a set

$$\underbrace{f^{-1}}_{\neq} (B) = \{ x \in X : f(x) \in B \} .$$

not inverse function.
[f could not be bijective] .

Ex: $f : N \to N$ s.t $y = 2x \ \forall x \in N$

- $A = \{1, 2, 3\}$
- $f(A) = \{2, 4, 6\}$
- $f^{-1}(A) = \{1\}$
- $(f \circ f^{-1})(A) = f(f^{-1}(\{1,2,3\}))$
$= f(1)$
$= 2 .$

This is not inverse function just inverse image.

- $(f^{-1} \circ f)(A) = f^{-1}(f(\{1,2,3\}))$
$= f^{-1}(\{2,4,6\})$
$= \{1, 2, 3\}$

Axiom 10 (Power set axiom):
Let X & Y be 2 sets ~~set~~ then $\exists$ another set

$$Y^X := \{ f : X \to Y \}$$

The functions are not considered as sets but as objects.

If $\#(Y)$ = Total element of Y $= n$
$\#(X)$ = " " " X then $= m$
$\#(Y^X) = (\#(Y))$ $= n^m$

Axiom 11:
Let A be a set which comprises objects that are themselves sets. Then
$\underline{UA}$ = set of objects which are elements of elements of A.
↳ union A.

- Power set of set X:
$P(X) = \{ Y : Y \subseteq X \}$

Claim: $f : X \to \{0, 1\}$
Axiom 10 $\{0,1\}^X = \{ f : X \to \{0,1\} \}$
I would like to show a bijective map $\{0,1\}^X \to P(X)$
$\psi : \{0,1\}^X \to P(X)$

To show $P(X)$ is a set under Axiom 5

for each $f \in \{0,1\}^x$ — ①
let $A_f = \{x : f(x) = 1\} \subseteq X$
and define $\psi(f) := A_f$

To prove : I) ψ is injective
Let $\psi(f) = \psi(g)$ $f, g \in \{0,1\}^X$
$\hookrightarrow A_f = A_g$
Goal: $f = g$ (To be proved)

① Let for $x \in X$ $f(x) = 0 \Rightarrow x \notin A_f \Rightarrow x \notin A_g \Rightarrow g(x) = 0$
② Let for $x \in X$ $f(x) = 1 \Rightarrow x \in A_f \Rightarrow x \in A_g \Rightarrow g(x) = 1$

$\therefore$ $f(x) = 0 \Rightarrow g(x) = 0$
$f(x) = 1 \Rightarrow g(x) = 1$
$\Rightarrow f(x) = g(x) \ \forall x$
$\Rightarrow f = g$
$\Rightarrow \psi$ is injective.

~~Through we can prove P(x) is a set~~

II) ψ is surjective
Let us define for some $A \subseteq X$

$$f_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \forall x \in X \quad \text{— ②}$$

characteristic func of set A.

It is clear that :
$f_A \in \{0,1\}^X$

$\psi(f_A) = \{x : f_A(x) = 1\}$
$= A$

f_A is coming from $A \subseteq X$

① For each $A \subseteq X$, we find f_A as in eq (2)
② For each $f \in \{0,1\}^X$ we find $A_f \subseteq X$ as in eq(1)

$\psi : \{0,1\}^X \to P(X)$ is bijective function.

Then from axiom 5 since $\{0,1\}^X$ is a set, ~~we~~ we have:
$\psi(\{0,1\}^X)$ is a set as well
$\Downarrow$
$P(X)$

- Cartesian Product :
- If X & Y are sets then $X \times Y$ is called cartesian product of X and Y and is defined as :

$$X \times Y = \{ (x, y) : x \in X, y \in Y \}$$

↑ ordered pair or 2-tuple

- n-fold cartesian products and n-tuples

n tuple : $(x_i)_{1 \le i \le n}$ or $(x_1, x_2 \ldots x_n)$

n-fold cart. Prod. of $x_1 \ldots x_n$: $\prod_{i=1}^{n} X_i$, $(X_i)_{1 \le i \le n}$

If all $x_i = x$ then $\prod_{i=1}^{n} X_i = x^n$

- Cardinality of sets :

Q) What is the size of a set ?

A) # of elements ($\# =$ no.) of

↳ problematic : for infinite sets

Def (equal cardinality) :

We say that sets X & Y have equal cardinality iff there exists a bijection from X to Y

↳ bijective func ↳ or $Y \to X$ as it is bijective.

Fun fact : N, $B = \{ 2n : n \in N \}$

$C = \{ 2n+1 : n \in N \}$

Q) Are cardinalities of N & B same?

A) $f: N \to B$ s.t.

$f(x) = 2x \quad \forall x \in N$

$f: B \to C$

$f(x) = 2x+1 \quad \forall x \in B$

$\Rightarrow$ B & C have same cardinality.

$\therefore$ N ~~has~~ & B have same cardinality.

$f: N \to A$ st

$f(x) = 2x+1 \quad \forall x \in W$

$\therefore$ N & A have same cardinality.

Here, $\left.\begin{array}{l} B \subseteq N \\ C \subseteq N \end{array}\right]$ But $\#B = \#N$ & $\#C = \#N$

$\therefore$ counting no. of elements is not the best way.

"Equal cardinality as a relation is an equivalence relation