§ Lecture 25.0

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Differentiability at a point :

Let $X \subseteq \mathbb{R}$. Let $x_0 \in X$ be a limit point of X.

Let $f: X \to \mathbb{R}$. If limit

$$\lim_{x \to x_0 \,;\, x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and is equal to L, then we say that f is differentiable at x_0 on X with derivative L.

$$f'(x_0) = L.$$

If limit doesn't exist

or if $x_0 \notin X$

or if x_0 is not a limit point of X

then f is not differentiable at x_0 on X.

[x_0 being a limit point is needed to define the limit]

Suppose $f(x) = x^2$ $x \in [1,2] \cup \{3\}$

then function is not differentiable at $x_0 = 3$ on $[1,2] \cup \{3\}$.

Ex 2 $f: \mathbb{R} \to \mathbb{R}$

$f(x) = |x|$ $x_0 = 0$

$$\lim_{x \to 0 \,;\, x \in \mathbb{R} - \{0\}} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0 \,;\, x \in \mathbb{R} - \{0\}} \frac{|x|}{x}$$

The right hand limit

$$\lim_{x \to 0 \,;\, x \in (0, \infty)} \frac{|x|}{x} = 1$$

Left hand limit

$$\lim_{x \to 0 \,;\, x \in (-\infty, 0)} \frac{|x|}{x} = -1$$

Thus $\displaystyle\lim_{x \to 0 \,;\, x \in \mathbb{R} - \{0\}} \frac{f(x) - f(0)}{x}$ doesn't exist.

And function is not differentiable at 0 on $\mathbb{R}$.

Suppose we restrict the domain of f to $[0, \infty)$

then

$$\lim_{x \to 0 \,;\, x \in [0,\infty) - \{0\}} \frac{f(x) - f(0)}{x} = \lim_{x \to 0 \,;\, x \in (0,\infty)} \frac{|x|}{x} = 1$$

Thus function $f|_{[0,\infty)}$ is differentiable at 0 on $[0, \infty)$ with derivative 1.

Differentiability implies continuity:

Let $X \subseteq \mathbb{R}$ and let $x_0 \in X$ be a limit point of X.

Let $f : X \to \mathbb{R}$ be a function. If f is differentiable at x_0 then f is also continuous at x_0.

Proof: Given that

$$\lim_{x \to x_0 \,;\, x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} := L \quad (\text{Exists})$$

Since x_0 is a limit point of X.

$\Rightarrow \exists$ a sequence $(x_n)_{n \geq 1}$ with $x_n \in X - \{x_0\}$ that converges to $x_0 \in X$.

Consider $\displaystyle f(x_n) - f(x_0) = \frac{f(x_n) - f(x_0)}{x_n - x_0} \times (x_n - x_0)$

$(x_n \neq x_0 \;\forall\, n \geq 1)$

$$\lim_{n\to\infty}\left(f(x_n)-f(x_0)\right) = \lim_{n\to\infty}\left(\frac{f(x_n)-f(x_0)}{x_n-x_0}\right)\lim_{n\to\infty}(x_n-x_0)$$

$$=0$$

$$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x_0)$$

$\Rightarrow$ f is continuous at x_0.

Proposition: (Newton's approximation)

Let $X\subseteq\mathbb{R}$. Let $x_0\in X$ be a limit point of X. Let $f: X\to\mathbb{R}$

be a function. and $L\in\mathbb{R}$. Then following statements are

equivalent.

(a) $f'(x_0)=L$.

(b) $\forall\, \varepsilon>0\ \exists\, \delta>0$ s.t.

$$\left|f(x)-\left(f(x_0)+L(x-x_0)\right)\right| \le \varepsilon|x-x_0|$$

whenever $x\in X$ and $|x-x_0|\le\delta$.

Proof: (a) $\Rightarrow$ (b)

Given $$\lim_{x\to x_0;\, x\in X-\{x_0\}} \frac{f(x)-f(x_0)}{x-x_0} = L$$

$\Rightarrow \forall\, \varepsilon>0\ \exists\, \delta>0$ s.t.

$$\left|\frac{f(x)-f(x_0)}{x-x_0}-L\right| \le \varepsilon \quad \text{whenever } x\in X-\{x_0\} \text{ and } |x-x_0|\le\delta.$$

$$\Rightarrow \left|f(x)-\left(f(x_0)+L(x-x_0)\right)\right| \le \varepsilon|x-x_0|.$$

For $x=x_0$, the inequality is trivially satisfied. Thus (b) is true.

(b) $\Rightarrow$ (a)

Since (b) is true, for a fixed $\varepsilon>0$ let $\delta>0$ be

such that $\forall\, x\in X$ with $|x-x_0|\le\delta$, we have

$$\left|f(x)-\left(f(x_0)+L(x-x_0)\right)\right| \le \varepsilon|x-x_0|$$

This is true for all $x \in X$ with $|x-x_0| \leq \delta$. We may

remove x_0, we still have

$$\left| f(x) - \left(f(x_0) + L(x-x_0) \right) \right| \leq \varepsilon |x-x_0| \quad \forall x \in X - \{x_0\}$$

with

$|x-x_0| \leq \delta$.

$$\Rightarrow \left| \frac{f(x)-f(x_0)}{x-x_0} - L \right| \leq \varepsilon \quad \text{whenever}$$

$|x-x_0| \leq \delta \;\; \forall \;\; x \in X - \{x_0\}$

$$\Rightarrow \lim_{x \to x_0;\, x \in X - \{x_0\}} \frac{f(x)-f(x_0)}{x-x_0} = L.$$

$$\Rightarrow \quad f(x) \approx f(x_0) + (x-x_0) f'(x_0)$$

(As long as $(x-x_0)$ is small).

§ Lecture 25.1

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Differential Calculus:

Premise: $X \subseteq \mathbb{R}$. $x_0 \in X$ be a limit point of X.

$f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ be two functions.

ⓐ If $f(x) = c \ \forall x \in X$ then $f'(x_0) = 0$.

ⓑ If $f(x) = x \ \forall x \in X$ then $f'(x_0) = 1$

ⓒ If f and g are differentiable at x_0 then

$(f \pm g)$ is also differentiable at x_0.

ⓓ If f, g are differentiable at x_0 then fg is also differentiable at x_0 and

$$(fg)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0)$$

ⓔ If g is differentiable at x_0 and $g(x) \neq 0 \ \forall x \in X$ then $1/g$ is also differentiable and

$$\left(\frac{1}{g}\right)'(x_0) = -\frac{g'(x_0)}{(g(x_0))^2}.$$

Proof: ⓐ $f(x) = c \ \forall x \in X$

$$\lim_{x \to x_0;\, x \in X - \{x_0\}} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0;\, x \in X - \{x_0\}} \frac{c - c}{x - x_0} = 0.$$

ⓓ
$$\frac{(fg)(x) - (fg)(x_0)}{x - x_0} = \frac{f(x)\, g(x) - f(x_0)\, g(x_0)}{x - x_0}$$

$$= \frac{(f(x) - f(x_0))\, g(x) + f(x_0)(g(x) - g(x_0))}{(x - x_0)}$$

$$\lim_{x \to x_0;\, x \in X - \{x_0\}} \frac{fg(x) - fg(x_0)}{x - x_0} = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0).$$

ⓔ
$$\frac{\frac{1}{g}(x) - \frac{1}{g}(x_0)}{} = \frac{g(x_0) - g(x)}{}$$

$$\frac{\quad}{x-x_0} = \frac{\quad}{(x-x_0)\, g(x)\, g(x_0)}$$

$$\lim_{x \to x_0;\, x \in X - \{x_0\}} \frac{\frac{1}{g}(x) - \frac{1}{g}(x_0)}{x - x_0} = -\frac{g'(x_0)}{(g(x_0))^2}$$

Chain rule:

Let $X, Y \subseteq \mathbb{R}$. Let $x_0 \in X$ and $y_0 \in Y$ be limit points of X and Y, respectively. Let $f : X \to Y$ be a function such that $f(x_0) = y_0$ and f is differentiable at x_0. Suppose $g : Y \to \mathbb{R}$ be a differentiable function at y_0. Then $g \circ f : X \to \mathbb{R}$ is differentiable at x_0 and

$$(g \circ f)'(x_0) = g'(y_0)\, f'(x_0).$$

Proof: We want to prove that

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$|(g \circ f)(x) - g \circ f(x_0) - AB(x - x_0)| \leq \varepsilon |x - x_0| \longrightarrow (1)$$

whenever $|x - x_0| \leq \delta$.

where $A = f'(x_0)$ $B = g'(y_0)$ $y_0 = f(x_0)$.

Fix some $\varepsilon > 0$.

Define $\varepsilon_1 = \min\left\{1, \frac{\varepsilon}{|A| + |B| + 1}\right\} \longrightarrow (2)$

Given since f is differentiable at x_0, we have

$$|f(x) - f(x_0) - A(x - x_0)| \leq \varepsilon_1 |x - x_0|$$

whenever $|x - x_0| \leq \delta_1$ (for some δ_1) $\longrightarrow (3)$

Note that

$$|f(x) - y_0| \leq |f(x) - y_0 - A(x - x_0)| + |A(x - x_0)|$$

$$\leq (|A| + \varepsilon_1)\, |x - x_0|$$

$\longrightarrow (4)$

whenever $|x - x_0| \leq \delta_1$

whenever $|x-x_0| \leq \delta_1$

Since g is differentiable at y_0

$\exists\ \delta_2 > 0$ s.t.

$$\left| g(f(x)) - g(y_0) - B(f(x)-y_0) \right| \leq \varepsilon_1 |f(x)-y_0|$$

whenever $|f(x)-y_0| \leq \delta_2$. $\longrightarrow$ (5)

Pick η s.t. $(|A|+\varepsilon_1)\eta \leq \delta_2$

$\delta = \min\{\delta_1, \eta\}$

Then $|x-x_0| \leq \delta \Rightarrow |x-x_0| \leq \delta_1$

and $|f(x)-y_0| \leq (|A|+\varepsilon)\eta \leq \delta_2$.

Then eq^n (5) is valid & eq^n (3).

We have

$|g\circ f(x) - g(y_0) - AB(x-x_0)|$

$= |g\circ f(x) - g(y_0) - B(f(x)-y_0) + B(f(x)-y_0) - AB(x-x_0)|$

$\leq \varepsilon_1 |f(x)-y_0| + |B|\ \varepsilon_1 |x-x_0|$

$\leq \varepsilon_1 (|A|+\varepsilon_1)|x-x_0| + |B|\varepsilon_1 |x-x_0|$

$= \varepsilon_1 (|A|+|B|+\varepsilon_1)|x-x_0|$

$\leq \varepsilon_1 (|A|+|B|+1)|x-x_0|$

$\leq \varepsilon |x-x_0|$

whenever $|x-x_0| \leq \delta$.

Thus $(f\circ g)'(x_0) = AB = f'(x_0)\, g'(y_0)$.