§ Lecture 24.0

Monday, 10 November 2025 23:50

Recall:

Heine-Borel theorem:

(a) X is closed and bounded.

(b) Given $(a_n)_{n\geq 0}$ with $a_n \in X$, then exists a subsequence $(a_{n_j})_{j\geq 0}$ that converges to some number in X.

Theorem (Intermediate value theorem):

Let $a<b \in \mathbb{R}$. $f:[a,b]\to\mathbb{R}$ be a continuous function on $[a,b]$. Let y be a real number between $f(a)$ and $f(b)$. Then $\exists\ c\in[a,b]$ s.t. $f(c)=y$.

Proof: Let $f(a)\leq y\leq f(b)$.

If $y=f(a)$ or $y=f(b)$, then we are done.

So assume $f(a)<y<f(b)$.

Define $F=\{x\in[a,b] : f(x)<y\}$

Since $a\in F$ as $f(a)<y$, F is non empty.

$F\subseteq[a,b]$

So it is upper bounded. Then from the principle of least upper bound $\sup(F)$ exists.

Let $c=\sup(F)$.

Goal: We would like to claim that

$c \in [a,b]$ and $f(c) = y$.

① Note that $a \in F \Rightarrow a \leq \sup(F) = c$.

② Now let $c > b$. Then b is smaller than c

and further b is an upper bound F.

$\Rightarrow$ c cannot be supremum.

$\Rightarrow$ $c \leq b$. [or since $c = \sup(F)$ and b is upper bound on $F \Rightarrow c \leq b$.]

Thus $c \in [a,b]$.

Now claim 1: $f(c) \leq y$. (Left of c)

For all $n \geq 1$ $c - \frac{1}{n} < c$

Then there exists $x_n \in F$ s.t. $x_n > c - \frac{1}{n}$

Further $c - \frac{1}{n} < x_n \leq c$

Taking limit $c \leq \lim_{n\to\infty} x_n \leq c$

or $\lim_{n\to\infty} x_n = c$

Since f is continuous on $[a,b]$

$\lim_{n\to\infty} f(x_n) = f(c)$

Since $x_n \in F$ $f(x_n) < y$

Taking limit both sides

$f(c) \leq y. \longrightarrow ①$

Now claim 2: $f(c) \geq y$ (right of c).

Since $f(a) < y < f(b)$

we have $f(c) \leq y < f(b)$

$\Rightarrow f(c) < f(b)$

$\Rightarrow c \neq b$ otherwise $f(c) = f(b)$.

Since $c \neq b$ and $c \in [a,b]$ $\Rightarrow c < b$.

Then $\exists\, N>0$ s.t. $c+\frac{1}{n}<b$ ...

But $c+\frac{1}{n} > c = \sup(F)$

If $c+\frac{1}{n} \in F \Rightarrow c$ cannot be an upper bound and hence supremum.

$\Rightarrow c+\frac{1}{n} \notin F \Rightarrow f(c+\frac{1}{n}) \geq y.$ —(2)

Let $b_n = c+\frac{1}{n} \Rightarrow \lim_{n\to\infty} b_n = c \in [a,b]$

and f is continuous $\Rightarrow \lim_{n\to\infty} f(b_n) = f(c)$

Taking limits on (2) $f(c) \geq y$.

Thus $f(c) = y$

Example: $f : [0,2] \to \mathbb{R}$

$f(x) = x^2$

$f(0)=0$; $f(2)=4$

Let $y=2$ s.t. $f(0) < y < f(2)$.

Then there exists $c \in [0,2]$ s.t. $f(c)=y$

or $c^2 = 2$.

That is there exists a real number $c \in [0,2]$ s.t. $c^2 = 2$

Uniform continuity:

Informal: $f : (0,2) \to \mathbb{R}$

$f(x) = \frac{1}{x}$

fix any $a \in (0,2)$. Fix any $\varepsilon > 0$.

We must find $\delta > 0$ s.t. $\forall\, x \in (0,2)$ with

$|x-a| < \delta$, we have $|f(x) - f(a)| \leq \varepsilon$

$$\left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{|x|\,|a|} \leq \varepsilon$$

Choose $\delta_1 = a/2$. If $|x-a| < \delta_1$

$$\Rightarrow x > a - \delta_1 = a/2$$

$$\Rightarrow |x| \geq a/2$$

$$\frac{1}{|x||a|} \leq 2/a^2$$

$$\Rightarrow \left|\frac{1}{x} - \frac{1}{a}\right| = \frac{|x-a|}{|x||a|} \leq \frac{2|x-a|}{a^2}$$

choose $\delta_2 = \frac{a^2}{2}\varepsilon$

If $|x-a| < \delta_2$ then $\frac{2|x-a|}{a^2} < \varepsilon$

Take $\delta = \min\{a/2, \frac{a^2\varepsilon}{2}\}$

then $|x-a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Choose $\varepsilon = 0.1$, $a = 1$ then

$$\delta = \min\{1/2, 0.1/2\} = 0.05$$

for $a = 0.1$, $\varepsilon = 0.1$

$$\delta = \min\{0.05, \frac{0.001}{2}\} = 0.0005$$

So for a fix ε, δ depends on a and ε.

Sequential def: Let (x_n) be a sequence converging to a.

$$\lim_{n\to\infty} x_n = a \qquad x_n \in (0,2).$$

Then

$$|f(x_n) - f(a)| = \frac{|x_n - a|}{|x_n||a|}$$

choose $\varepsilon = a/2$ $\exists N > 0$ s.t $\forall n \geq N$

$$|x_n - a| \leq \varepsilon = a/2$$

$$a/2 \leq x_n \leq 3a/2$$

Since $x_n \geq a/2 \Rightarrow |x_n| \geq a/2$

Then $|f(x_n) - f(a)| \leq \frac{2}{a^2}|x_n - a|$

$$\lim |f(x_n) - f(a)| \leq \lim \frac{2}{a^2}|x_n - a|$$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} |f(x_n) - f(a)| = 0$$

$\Rightarrow \forall \varepsilon > 0 \quad \exists N > 0$ s.t. $\forall n \geq N$

$$\big|\, |f(x_n) - f(a)| - 0 \,\big| \leq \varepsilon \,.$$

or $\quad |f(x_n) - f(a)| \leq \varepsilon$

$$\Rightarrow \quad \lim_{n\to\infty} f(x_n) = f(a).$$

Is it bounded?

Fix $M > 0$.

choose $x = \frac{1}{M+1}$

Then $x \in (0,2)$

$$f(x) = M+1 > M$$

Thus $\forall M > 0 \quad \exists x$ s.t. $f(x) > M$

$\Rightarrow f$ is not bounded.

f is bounded from below as $f(x) \geq 0 \;\; \forall\, x \in (0,2)$.

§ Lecture 24.1

Wednesday, 12 November 2025 21:33

Uniform continuity : Let $X \subseteq \mathbb{R}$ and $f: X \to \mathbb{R}$ be a function. We say that f is uniformly continuous if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x, x_0 \in X$ with

$|x - x_0| < \delta$, we have $|f(x) - f(x_0)| \leq \varepsilon$.

Importantly δ doesn't depend on x_0.

However, it depends on ε only.

$f(x) = 1/x$ $\quad x \in (0,2)$ is not uniformly continuous but continuous.

$f(x) = 2x$

Let $|x - a| < \delta$

$|f(x) - f(a)| = 2|x-a|$

Take $\delta = \varepsilon/2$

Then $\forall \varepsilon > 0$ $\quad |x-a| < \varepsilon/2 \Rightarrow |f(x) - f(a)| < \varepsilon$

$\forall a \in (0,2)$

Theorem: Any continuous function $f: [a,b] \to \mathbb{R}$ $(a<b)$ is uniformly continuous.

Limits at infinity:

Def: Let $X \subseteq \mathbb{R}$. We say that $+\infty$ $(-\infty)$ is an adherent point to X iff $\forall M \in \mathbb{R}$ $\exists x \in X$ s.t. $x > M$ $(x < M)$.

Def: Let $X \subseteq \mathbb{R}$ and $+\infty$ be an adherent point of X.

Then $\displaystyle \lim_{x \to \infty;\, x \in X} f(x) = L$ iff

$\forall \varepsilon > 0$ $\exists M$ s.t. $|f(x) - L| \leq \varepsilon$ $\forall x \in X$ s.t. $x > M$

or $x \in X \cap (M, \infty)$

Similarly $\lim_{x \to -\infty;\, x \in X} f(x) = L$ iff

$\forall \varepsilon > 0 \ \exists M$ s.t. $|f(x) - L| \leq \varepsilon \ \forall x \in X \cap (-\infty, M)$.

Ex: $\lim_{x \to \infty} \frac{1}{x}$ $\quad X = (0, \infty)$

$f : X \to \mathbb{R} \quad f(x) = 1/x$.

fix some $\varepsilon > 0$.

choose $M = 1/\varepsilon$ $\quad x > M \Rightarrow \frac{1}{x} < \frac{1}{M}$

$|f(x) - 0| = \frac{1}{x} < \frac{1}{M} = \varepsilon$

Yes.

Proposition: Let $A, B \subseteq \mathbb{R}$.

Let $g : A \to \mathbb{R}$ and $f : B \to \mathbb{R}$ be functions with

$g(A) \subseteq B$. Let $c \in A$. If

(i) g is continuous at c

(ii) f is continuous at $g(c) \in B$, then

$f \circ g : A \to \mathbb{R}$ is continuous at c.

Proof: suppose c is finite.

Let $(x_n) \subset A$ be a sequence with

$\lim_{n \to \infty} x_n = c$

Then since g is continuous

$\lim_{n \to \infty} g(x_n) = g(c)$

That is sequence $(g(x_n)) \subseteq B$ converges to $g(c)$.

But f is continuous at $g(c)$ then

$\lim_{n \to \infty} f(g(x_n)) = f(g(c))$

$\Rightarrow$ $f \circ g$ is continuous at c.

Infinite case:

Let $U, X \subseteq \mathbb{R}$. Let $+\infty$ be adherent to X.

$f: U \to \mathbb{R}$ $\quad g: X \to \mathbb{R}$ and $g(X) \subset U$.

Let $\lim_{x \to \infty;\, x \in X} g(x) = L$

and f is continuous at L. Then

$$\lim_{x \to \infty;\, x \in X} f(g(x)) = f(L).$$

Proof: Fix $\varepsilon > 0$. Since f is continuous at L.

$\Rightarrow$ $\exists\, \eta_0 > 0$ such that for $y \in U$ wins

$$|y - L| < \eta_0 \Rightarrow |f(y) - f(L)| < \varepsilon. \longrightarrow (1)$$

Since $\lim_{x \to \infty,\, x \in X} g(x) = L$

$\exists\, M$ s.t. $\forall x \in X \cap (M, \infty)$ $\quad |g(x) - L| < \eta$ $\forall \eta > 0$.

Fix η_0

Then $\forall x \in X \cap (M, \infty)$ $\quad |g(x) - L| < \eta_0$ $\quad (g(x) \in U.)$

$\Rightarrow$ $|f(g(x)) - f(L)| < \varepsilon$ $\quad \forall \varepsilon > 0$.

$\Rightarrow$ $\exists\, M$ s.t. $\forall x \in X \cap (M, \infty)$

$$|f \circ g(x) - f(L)| < \varepsilon \quad \forall \varepsilon > 0.$$

$$\lim_{x \to \infty} f \circ g(x) = f(L) = f\left(\lim_{x \to \infty} g(x)\right)$$

Ex: $f: \mathbb{R} \to \mathbb{R}$ with $f(x) = x\left(\sqrt{1+x^2} - x\right)$

What is $\lim_{x \to \infty} f(x)$?

Define $g: \mathbb{R} - \{0\} \to \mathbb{R}$ by $\quad\Big|\quad f(x) = \dfrac{1}{1 + \sqrt{1 + 1/x^2}}$

$g(x) = \frac{1}{x}$

$f : [-1, \infty] \to \mathbb{R}$

$f(t) = \dfrac{1}{1 + \sqrt{1+t}}$

$\lim_{x \to \infty} g(x) = 0$

and f is continuous at 0.

Then $\lim_{x \to \infty} f(g(x)) = f(0) = \frac{1}{2}$.