§ Lecture 23.0

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Proposition (Equivalent definitions of continuity):

Let $X \subseteq \mathbb{R}$. $f: X \to \mathbb{R}$ be a function. $x_0 \in X$. Then following statements are equivalent.

1. f is continuous at x_0.
2. For every sequence $(a_n)_{n \geq 0}$ where $a_n \in X$ with $\lim_{n\to\infty} a_n = x_0$, $\lim_{n\to\infty} f(a_n) = f(x_0)$.
3. For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ $\forall x \in X$ with $|x - x_0| < \delta$.
4. For every $\varepsilon > 0$, $\exists$ a $\delta > 0$ s.t. $|f(x) - f(x_0)| \leq \varepsilon$ $\forall x \in X$ with $|x - x_0| \leq \delta$.

Proof: 1 $\Leftrightarrow$ 2 (From sequential version of limits)

2 $\Rightarrow$ 3

Assume 2. Suppose 3 is false.

Then $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$ $\exists x \in X$ with $|x - x_0| < \delta$ but $|f(x) - f(x_0)| \geq \varepsilon_0$.

In particular for $\delta = 1/(n+1)$, choose $x_n \in X$ s.t.

$$|x_n - x_0| < \frac{1}{n+1} \quad \text{and}$$

$$|f(x_n) - f(x_0)| \geq \varepsilon_0$$

Then $x_n \to x_0$
but $f(x_n) \not\to f(x_0)$

contradicts 2.

$\Rightarrow$ 3 is true.

3 $\Rightarrow$ 2

Assume ③. Let $a_n \in X$ and $(a_n)_{n\geq 0}$ be a sequence with $a_n \to x_0$. Fix $\varepsilon > 0$. From ③

$\exists \delta > 0$ s.t. $\forall x \in X$ $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Since $a_n \to x_0$ we have

$$|a_n - x_0| < \delta \quad \forall n \geq N.$$

Thus for $n \geq N$ $|f(a_n) - f(x_0)| < \varepsilon$.

$$\Rightarrow f(a_n) \to f(x_0).$$

③ $\Rightarrow$ ④ $\forall \varepsilon > 0$ $\exists \delta' > 0$

$|x - x_0| < \delta' \Rightarrow |f(x) - f(x_0)| < \varepsilon$

choose $\delta = \delta'/2$

Take any $x \in X$ with $|x - x_0| \leq \delta$.

Then $|x - x_0| \leq \delta < \delta' \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\Rightarrow |f(x) - f(x_0)| \leq \varepsilon$.

④ $\Rightarrow$ ③ $\quad$ ($\{x : |x - x_0| \leq \delta\} \subset \{|x - x_0| < \delta'\}$.

Fix $\varepsilon > 0$. Apply ④ for $\varepsilon' = \varepsilon/2$

$\exists \delta > 0$ s.t.

$|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon'$

Take any x with $|x - x_0| \leq \delta \Rightarrow |f(x) - f(x_0)| \leq \varepsilon' < \varepsilon$.

Claim (Arithmatic preserve continuity)

$f, g : X \to \mathbb{R}$ continuous at x_0. Then

① $f \pm g$

② f/g if $g(x) \neq 0$ $\forall x \in X$ etc. are continuous at x_0.

<u>Left and right limits</u>:

$X \subset \mathbb{R}$ $f : X \to \mathbb{R}$ be a function $x \in \mathbb{R}$ at x_0

$X \subseteq \mathbb{R}$. $f: X \to \mathbb{R}$ be a function. $x_0 \in \mathbb{R}$. If x_0

is an adherent point of $X \cap (x_0, \infty)$ then

right limit $f(x_0+)$ of f at x_0 is defined as

$$f(x_0+) = \lim_{x \to x_0;\, x \in X \cap (x_0, \infty)} f(x). = \lim_{x \to x_0^+} f(x)$$

If x_0 is an adherent point of $X \cap (-\infty, x_0)$, then

left limit $f(x_0-)$ of f at x_0 is defined as

$$f(x_0-) = \lim_{x \to x_0;\, x \in X \cap (-\infty, x_0)} f(x). = \lim_{x \to x_0^-} f(x)$$

<u>Proposition</u>: Let $X \subseteq \mathbb{R}$. $x_0 \in \mathbb{R}$ and let x_0 be an

adherent point of both $X \cap (x_0, \infty)$ and $X \cap (-\infty, x_0)$.

If $f(x_0+)$ and $f(x_0^-)$ both exists and are equal to $f(x_0)$

then $f(x)$ is continuous at x_0.

<u>Proof</u>:

We need to prove that $\forall\, \varepsilon > 0\ \exists\, \delta$ s.t.

$\forall\, x \in X$ with $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Given

$$\lim_{x \to x_0^+} f(x) = f(x_0) = \lim_{x \to x_0^-} f(x).$$

$$\Downarrow$$

$\forall\, \varepsilon > 0\ \exists\, \delta_+ > 0$ s.t. $\forall\, x \in X \cap (x_0, \infty)$

if $|x - x_0| < \delta_+ \Rightarrow |f(x) - f(x_0)| < \varepsilon$. $\rightarrow (1)$

$\forall\, \varepsilon > 0\ \exists\, \delta_- > 0$ s.t. $\forall\, x \in X \cap (-\infty, x_0)$

if $|x - x_0| < \delta_- \Rightarrow |f(x) - f(x_0)| < \varepsilon$. $\rightarrow (2)$

Define $\delta = \min\{\delta_+, \delta_-\} > 0$

Consider an arbitrary $x \in X$ with $|x - x_0| < \delta$.

Then

Case 1 $x > x_0$

$x \in X \cap (x_0, \infty)$

Since $|x - x_0| < \delta \leq \delta_+$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ (From ①)

Case 2 $x < x_0$

Then $x \in X \cap (-\infty, x_0)$

Since $|x - x_0| < \delta \Rightarrow |x - x_0| < \delta_-$

$\Rightarrow |f(x) - f(x_0)| < \varepsilon$ (From 2)

Case 3: $x = x_0$

Then $|x - x_0| = 0 < \delta$

and $|f(x) - f(x_0)| = 0 < \varepsilon$.

Trivially satisfied.

Thus
$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in X$ with $|x - x_0| < \delta$

$|f(x) - f(x_0)| < \varepsilon$.

$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0)$ and function is continuous at x_0.

The converse is true as well.

§ Lecture 23.1

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Bounded functions: $X \subseteq \mathbb{R}$ $f: X \to \mathbb{R}$

f is bounded from above if $\exists\, M \in \mathbb{R}$ s.t.

$$f(x) \leq M \quad \forall\, x \in X.$$

f is bounded from below if $\exists\, M \in \mathbb{R}$ s.t.

$$f(x) \geq -M \quad \forall\, x \in X.$$

f is bounded if $\exists\, M \in \mathbb{R}$ s.t.

$$|f(x)| \leq M \quad \forall\, x \in X.$$

Lemma: Let $a<b$ be real numbers. $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f is a bounded function.

Proof: Suppose f is not bounded. Thus for every real number M $\exists\, x \in [a,b]$ s.t.

$$|f(x)| \geq M.$$

In particular ~~take~~ $M = n \in \mathbb{N}$. Then

$$S_n = \{ x \in [a,b] \ . \ |f(x)| \geq n \} \quad \forall n.$$

is nonempty. Let $x_n \in S_n$.

Thus we can choose a sequence $(x_n)_{n \geq 0}$

s.t. $|f(x_n)| \geq n \ . \ \forall n \in \mathbb{N}.$

The sequence (x_n) lies in $[a,b]$ so it is closed and bounded.

From Heine-Borel theorem there exists a subsequence $(x_{n_j})_{j=0}^{\infty}$ converges to some limit $L \in [a,b]$, where $n_0 < n_1 \cdots$

Also $n_j \geq j \ \forall\, j \in \mathbb{N}$ [Induction $n_0 \geq 0$]

$n_j \geq j$

$n_{j+1} > n_j \geq j$

$\Rightarrow n_{j+1} > j$

$\Rightarrow n_{j+1} \geq j+1$

Since f is continuous on $[a,b]$, it is continuous at L. Thus

$$x_{n_j} \to L \Rightarrow \lim_{j\to\infty} f(x_{n_j}) = f(L)$$

Thus sequence $(f(x_{n_j}))_j$ is convergent.

$\Rightarrow (f(x_{n_j}))_j$ is bounded.

On Other hand we know that

$$|f(x_{n_j})| \geq n_j \geq j$$

That it is not bounded. A contradiction.

<u>Definition</u>: Let $f: X \to \mathbb{R}$ be a function. Let $x_0 \in X$. We say that f attains its maximum (minimum) at x_0 if $f(x_0) \geq f(x) \; \forall x \in X$

i.e. $(f(x_0) \leq f(x) \; \forall x \in X)$.

Proposition: Let $a<b$ be real numbers and let $f: [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Then f attains its maximum at some point $x_{max} \in [a,b]$ and minimum at some point $x_{min} \in [a,b]$.

<u>Nontriviality</u>: $f: (0,1) \to \mathbb{R}$ defined by

$f(x) = x$.

$\sup f(x) = 1$ but no point $(0,1)$ satisfy $f(x) = 1$.

Proof. We will prove for maximum only.

Since f is continuous on $[a,b]$, it is bounded. $\exists M$ s.t.

$$|f(x)| \leq M \quad \forall x \in [a,b]$$

Let $E = \{ f(x) : x \in [a,b] \}$

$= f([a,b])$ [Non empty]

This set is subset of $[-M, M]$.

Then from the least upper bound principle it has a supremum $\sup(E) \in \mathbb{R}$.

Let $m = \sup(E)$

That is $\forall y \in E$

$y \leq m$.

ie. $f(x) \leq m \quad \forall x \in [a,b]$.

To show that f attains maximum, it suffices to find $x_{max} \in [a,b]$ s.t. $f(x_{max}) = m$.

Let $n \geq 1$ be any integer. Then

$$m - \frac{1}{n} < m = \sup(E)$$

$\Rightarrow m - \frac{1}{n}$ cannot be an upper bound on E.

Thus $\exists y \in E$ s.t. $y > m - \frac{1}{n}$

That is $\exists x \in [a,b]$ s.t

$$f(x) > m - \frac{1}{n}.$$

Now choose a sequence $(x_n)_{n \geq 1}$ by choosing $x_n \in [a,b]$ s.t. $f(x_n) > m - \frac{1}{n}$.

[Requires axiom of choice]
Guarantees simultaneous
choices of x_n $\forall n \geq 1$.

This is a sequence in $[a,b]$.

From Heine-Borel theorem, we can find a subsequence

$(x_{n_j})_{j\geq 1}$ where $n_1 < n_2 \cdots$ which converges to $x_{max} \in [a,b]$

Now since f is continuous at x_{max}

$$\lim_{j\to\infty} f(x_{n_j}) = f(x_{max}).$$

On the other hand

$$f(x_{n_j}) > m - \frac{1}{n_j} \geq m - \frac{1}{j} \quad (n_j \geq j)$$

$$\Rightarrow \lim_{j\to\infty} f(x_{n_j}) \geq m - \lim_{j\to\infty} \frac{1}{j}$$

or $\quad f(x_{max}) \geq m$

But $\quad f(x_{max}) \leq m$

$\Rightarrow \quad f(x_{max}) = m.$

[x_{max} is not unique]

e.g. $\quad f(x) = x^2 \quad$ in $[-2,2]$

$x_{max} = -2, 2.$