§ Lecture 20.0

Thursday, 16 October 2025 20:32

Q. $\overline{Q} = \mathbb{R}$.

$\overline{Q} \subseteq \mathbb{R}$ (Trivial)

$x \in \overline{Q} \Rightarrow \forall \varepsilon > 0 \ \exists z \in Q$ s.t.

$|z - x| \leq \varepsilon$

or $-\varepsilon \leq z - x \leq \varepsilon$

or $z - \varepsilon \leq x \leq z + \varepsilon$

or $x \in \underbrace{[z-\varepsilon, z+\varepsilon]}_{\text{Real interval}}$

$\Rightarrow x \in \mathbb{R}$

$\Rightarrow \overline{Q} \subseteq \mathbb{R}$.

$\mathbb{R} \subseteq \overline{Q} \Leftrightarrow x \notin \overline{Q} \Rightarrow x \notin \mathbb{R}$.

$x \notin \overline{Q} \Rightarrow \exists \varepsilon > 0$ s.t. there exists no

$z \in Q$ with

$|x - z| \leq \varepsilon$

or $\nexists$ no $z \in Q$ s.t.

$x - \varepsilon \leq z \leq x + \varepsilon$

or $[x-\varepsilon, x+\varepsilon] \cap Q = \phi$

However if $x \in \mathbb{R}$ ($\varepsilon \in \mathbb{R}$) then we know that there exists a z such that $x - \varepsilon \leq z \leq x + \varepsilon$.

$\Rightarrow x \notin \mathbb{R}$

or $\mathbb{R} \subseteq \overline{Q}$

Thus $\mathbb{R} = \overline{Q}$.

Ex. $(1,2) \cap \{3\} = X$

① 1, 2, 3 are limit points.

(1a) 1 is a limit point of X.

Assume 1 is not a limit point of X.

$\Rightarrow \exists \varepsilon > 0$ s.t. $\forall z \in X - \{1\} = X$

$|z-1| > \varepsilon$

or $[1-\varepsilon, 1+\varepsilon] \cap X = \phi \longrightarrow (1)$

Let $\delta = \min\{1, \varepsilon\}/2$

(1) $0 < \delta \leq 1/2 < 1$

(2) $0 < \delta \leq \varepsilon/2 < \varepsilon$

$1 < 1+\delta < 1+\varepsilon \Rightarrow 1+\delta \in (1, 1+\varepsilon)$

Also $1+\delta \in [1-\varepsilon, 1+\varepsilon]$

Further $1 < 1+\delta < 2 \Rightarrow (1+\delta) \in (1, 2)$

$\Rightarrow (1+\delta) \in [1-\varepsilon, 1+\varepsilon] \cap (1, 2)$

$\in [1-\varepsilon, 1+\varepsilon] \cap X$

$\Rightarrow [1-\varepsilon, 1+\varepsilon] \cap X \neq \phi \longrightarrow (2)$

Contradicting (1).

(2) Same for $x = 2$.

Suppose 2 is not limit point of X.

$\Rightarrow [2-\varepsilon, 2+\varepsilon] \cap Q = \phi$

But $2 > 2-\delta > 2-\varepsilon \Rightarrow 2-\delta \in [2-\varepsilon, 2+\varepsilon]$

Further $1 < 2-\delta < 2 \Rightarrow 2-\delta \in (1, 2)$

$\Rightarrow 2-\delta \in [2-\varepsilon, 2+\varepsilon] \cap (1, 2)$

or $2-\delta \in [2-\varepsilon, 2+\varepsilon] \cap X$

(3) 3 is not limit point.

As 3 is not an adherent point of $X - \{3\} = (1, 2)$

... is an isolated point as $3 \in X$.

Because $\forall y \in (1,2)$

$$|3-y| > 1$$

That there exists $\varepsilon = 1$ s.t. $\forall y \in (1,2)$

$$|3-y| > 1.$$

Proposition: Let $X \subseteq \mathbb{R}$. Let X' be the set of all limit points. Then

$$\overline{X} = X \cup X'.$$

Proof: $\overline{X}$ = set of all adherent points.

(1) $\Rightarrow$ $X \cup X' \subseteq \overline{X}$

Let $x \in X \cup X'$

(1a) let $x \in X \Rightarrow \forall \varepsilon > 0 \ \exists y = x \in X$ s.t.
$|y - x| \leq \varepsilon$.
$\Rightarrow x \in \overline{X}$

(1b) let $x \in X' \Rightarrow \forall \varepsilon > 0 \ \exists y \in X - \{x\}$ s.t.
$|y - x| \leq \varepsilon$.

This $y \in X \Rightarrow x \in \overline{X}$

$\Rightarrow X \cup X' \subseteq \overline{X}$.

(2) $\Leftarrow$ Let $x \in \overline{X} \Rightarrow x \in X \cup X'$.

(2a) If $x \in X' \Rightarrow x \in X \cup X' \Rightarrow \overline{X} \subseteq X \cup X'$

(2b) If $x \notin X' \Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall y \in X - \{x\}$
$|y - x| > \varepsilon_0$

Since $x \in \overline{X}$ we have that $\exists y \in X$ s.t.
$|y - x| \leq \varepsilon_0$.

This can happen only when $y = x$

$\Rightarrow x \in X$
$\Rightarrow x \in X \cup X'$

$\overline{X} \subseteq X \cup X'$

or $\boxed{\overline{X} = X \cup X'.}$ Standard definition.

§ Lecture 20.1

Friday, 24 October 2025 01:39

Proposition: x is a limit point of $X \subseteq \mathbb{R}$ iff $\exists$ a sequence $(a_n)_{n\geq 0}$, consisting of elements from $X-\{x\}$, such that $\lim_{n\to\infty} a_n = x$.

Proof: $\Rightarrow$ Let x be a limit point of X.

This implies $\forall\, \varepsilon > 0$ $\exists\, y \in X-\{x\}$ s.t.

$$|y-x| \leq \varepsilon.$$

Or for $\varepsilon_n = \frac{1}{n+1}$ $\exists\, a_n \in X-\{x\}$ s.t.

$$|x-a_n| \leq \frac{1}{n+1} \quad \forall n \geq 0.$$

Indeed such a sequence satisfies

$$x - \frac{1}{n+1} \leq a_n \leq x + \frac{1}{n+1}$$

$$x \leq \lim_{n\to\infty} a_n \leq x$$

or $\lim_{n\to\infty} a_n = x$.

$\Leftarrow$ Let $(a_n)_{n\geq 0}$ be a sequence of elements from $X-\{x\}$ s.t. $\lim_{n\to\infty} a_n = x$.

$\Rightarrow \forall\, \varepsilon > 0$ $\exists\, N_0 \geq 0$ s.t. $\forall n \geq N_0$

$$|a_n - x| \leq \varepsilon$$

In particular choose any $n \geq N_0$ with

$$a_{N_0} \in X-\{x\} \text{ s.t.}$$
$$|a_{N_0} - x| \leq \varepsilon.$$

$\Rightarrow x$ is a limit point of X.

Bounded sets: A set $X \subseteq \mathbb{R}$ is said to be bounded iff $\exists\, M > 0$ s.t. $X \subset [-M, M]$.

Heine-Borel theorem: Let $X \subseteq \mathbb{R}$. Then following statements are equivalent

① X is closed and bounded.

② Given any sequence $(a_n)_{n\geq 0}$ with $a_n \in X$, there exists a subsequence $(a_{n_j})_{j\geq 0}$ that converges to some $L \in X$.

Proof:

① ⇒ ②

Assume X is closed and bounded.

Let $(a_n)_{n\geq 0}$ be a sequence with $a_n \in X$.

Since X is bounded (a_n) is bounded sequence.

From Bolzano-Weierstrass theorem there is a subsequence (a_{n_j}) that converges to $L \in \mathbb{R}$.

Since (a_{n_j}) is a sequence of elements of X that converges to L, implies L is an adherent point of X, i.e. $L \in \overline{X} = X$ (closed).

② ⇒ ①.

Boundedness: Suppose X is unbounded. Then for each n we can pick $x_n \in X$ s.t. $|x_n| > n$.

From ② (x_{n_j}) is a convergent subsequence of x_n.

⇒ (x_{n_j}) is bounded. A contradiction.

⇒ X is bounded.

Closeness: Suppose X is not closed.

⇒ $\exists$ an adherent point x s.t. $x \notin X$.

Since x is an adherent point there exists a sequence

$(b_n) \subset X$ s.t. $b_n \to x$.

By ② there exists a subsequence (b_{n_j}) converging

to $L \in X$.

But $\lim\limits_{n\to\infty} b_n = \lim\limits_{j\to\infty} b_{n_j} = L = x$.

$\Rightarrow x \in X$. Contradiction

$\Rightarrow$ X is closed.