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•) Cancellation law :

If $m, n, l \in N$ then
$m+n = m+l \Rightarrow n = l$

Proof:

a) $P(m)$: $m+n = m+l$

b) $P(0)$: $0+n = 0+l \Rightarrow n = l$

c). Assume $P(m)$ is true, i.e.

$m+n = m+l \Rightarrow n = l$

$\Rightarrow$ show that $P(m_{++})$ is also true

•) Order relation :

• Let $m, n \in N$ then m is greater than equal to n ($m \geq n$ or $n \leq m$) if $\exists$ another natural number l such that

$m = n + l$

• m strictly greater than n
$m > n$ if $m \neq n$

• Trichotomy of N :

For any 2 $m, n \in N$, one of the relations is true
($m > n$, $n > m$, $n = m$)

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•) <u>Principle of well ordering</u> :

Every non-empty subset of N has a least element

<u>Proof</u> : [Proof by contradiction]

• Assume $\exists$ is a non-empty set S of N with no least element

a) Define property : $P(n)$: $n \notin S$ $\forall n \in N$

b) $P(0)$:

Suppose $0 \in S$ then it will least element. Then $0 \notin S$ by assumption.

contradict Assumption 1.

$\Rightarrow P(0)$ is true.

c) Let $P(n)$ is true i.e. $n \notin S$

we need to show that $P(n{+}{+})$ is true as well i.e. $n+1 \notin S$

If $n+1 \in S$ then $(n+1)$ will be a least element

$\Rightarrow P(n{+}{+})$ is true as well.

$\Rightarrow$ Initial Assumption is wrong.

•) <u>Multiplication</u>

<u>Def</u>:

a) $0 \times m = 0$

b) If $n \times m$ is defined then

$(n{+}{+}) \times m = n \times m + m$

<u>Claims</u>:

① $m \times 0 = 0$

② $m \times (n+l) = m \times n + m \times l$

③ $m \times n = n \times m$

•) Exponentiation :

a) $m^0 = 1$ [convention $0^0 = 1$]

b) If m^n is defined then
$m^{n+1} = m^n \times m$

•) Zermelo - Frankel set Theory :

• collection of elements / objects.

• 11 axioms.

• Axiom 1 : The sets are also objects.

• Axiom 2 : There exists an empty set ϕ that has no objects.

• Axiom 3 : Let a be an element / object then $\{a\}$ ~~is called~~ exists and is called a singleton set

$\Rightarrow \{\phi\}$ is a singleton set
$\{\phi\} \neq \phi$.

• Axiom 4 : If A & B are sets then $A \cup B$ is also a set which is defined as :

$A \cup B = \{ x : x \in A \text{ or } x \in B \}$

• $A = B \iff$ If $x \in A \Rightarrow x \in B$ & If $x \in B \Rightarrow x \in A$
$A \subseteq B$ $B \subseteq A$

• Axiom 5 | Axiom of specification :

Let A be a set and $x \in A$. Let $P(x)$ be a property of x. Then $\exists$ a set

$B = \{ x \in A \mid P(x) \text{ is true} \}$ [$B \subseteq A$ technically]

such that

$y \in B \Rightarrow P(y)$ is true

- Axiom 6 | Axiom of replacement:

Let A be a set. Let $x \in A$

Let y be an object

Let P(x, y) be a prop. such that for each $x \in A$,

there is at most one y such that

P(x, y) is true. Then ∃ a set:

$$B = \{ y \mid P(x,y) \text{ is true for some } x \in A \}$$

such that,

$$z \in B \Rightarrow P(x, z) \text{ is true for some } x \in A.$$

B: Range

This is essentially talking about functions.

- Axiom 7 | Axiom of Infinity:

∃ a set $\mathbb{N}$, called the set of natural numbers with objects $0 \in \mathbb{N}$ & $n{+}{+} \in \mathbb{N}$ for each $n \in \mathbb{N}$ satisfying all Peano axioms

- Axiom 8 | Universal specification: XX [FALSE. Look at Russel's Paradox]

Let x be an object and P(x) be a property of x.

Then ∃ a set

$$B = \{ x \mid P(x) \text{ is true} \}$$

s.t.

$$y \in B \Rightarrow P(y) \text{ is true}$$

Axiom 5 & 8 are similar. In 5, $x \in A$ & in 8 x is any object (not a part of a set).

- Russel's Paradox:

Property: $P(x)$: " x is a set and $x \notin x$ "

set & object

object

Define : $U = \{ x \mid P(x) \text{ is true} \}$ [Exists from axiom 8]

Q) $U \in U$?

set U is used → A) ① Say $U \in U \Rightarrow P(U)$ is true $\Rightarrow U$ is a set and $U \notin U$.

P(x) is used here → ② Say $U \notin U$ a) $\Rightarrow P(U)$ is true $\Rightarrow U \in U$

definition of set U is used → b) $\Rightarrow P(U)$ is false $\Rightarrow U \in U$

- Axiom 9 | Regularity:

If A is a non-empty set then there exists at least one element which is either not a set or ~~not equal~~ is disjoint to A.