§ Lecture 19.0

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Sequences are functions from $\mathbb{Z} \to \mathbb{R}$

$$n \to a_n$$

Now we will talk about functions on continuum.

such as real line $\mathbb{R}$.

Intervals : Let $a, b \in \mathbb{R}$

Closed intervals :-

$$[a,b] = \{x \in \mathbb{R}^* : a \leq x \leq b\}$$

Half open intervals :

$$[a,b) = \{x \in \mathbb{R}^* : a \leq x < b\}$$

$$(a,b] = \{x \in \mathbb{R}^* : a < x \leq b\}$$

Open intervals :

$$(a,b) = \{x \in \mathbb{R}^* : a < x < b\}$$

e.g.

$(0, +\infty)$ = Positive real axis $= \{x \in \mathbb{R} : x > 0\}$

$[0, +\infty)$ = Non negative real axis $= \{x \in \mathbb{R} : x \geq 0\}$

$(-\infty, +\infty)$ = Real line

$[-\infty, +\infty]$ = Extended real line

Adherent points of a subset: Let X be a subset of $\mathbb{R}$.

Let $x \in \mathbb{R}$. We say that x is an adherent point of X

iff $\exists\, y \in X$ s.t. $|x-y| \leq \varepsilon \quad \forall \varepsilon > 0$.

Example: Let $X = (0,1)$

Claim: 1 is adherent point of X.

Proof: Let $\varepsilon > 0$. We must find $y \in X$ s.t.

$|y-1| \leq \varepsilon.$

Let $y = 1-\varepsilon/2$

Case 1: If $0<\varepsilon<2$. Then $0<y<1 \Rightarrow y \in (0,1)$

Case 2: If $\varepsilon \geq 2$ Then $|y-1| < 1 \leq \varepsilon$ $\forall y \in (0,1)$

e.g. take $y=1/2$

$|1-1/2| = 1/2 \leq 2 \leq \varepsilon.$

$\Rightarrow \forall \varepsilon>0 \; \exists y \in X$ s.t. $|y-1| \leq \varepsilon.$

Closure of a set: Let $X \subseteq \mathbb{R}$. The closure of X, denoted as $\overline{X}$, is defined as the set of all adherent points of X.

Lemma: Let $X, Y \subseteq \mathbb{R}$.

(1) $X \subseteq \overline{X}$ (2) $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$

(3) $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$ (4) $X \subseteq Y \Rightarrow \overline{X} \subseteq \overline{Y}$

Proof:

(1) Let $x \in X$. For every $\varepsilon>0$ choose $y=x \in X$.

Then $|y-x| = 0 \leq \varepsilon. \Rightarrow x \in \overline{X}.$

$\Rightarrow X \subseteq \overline{X}.$

(2) $\Rightarrow$ Let $z \in \overline{X \cup Y}.$

Then $\forall \varepsilon>0 \; \exists w \in X \cup Y$ s.t. $|z-w| \leq \varepsilon.$

Assume $z \notin \overline{X}$ and $z \notin \overline{Y}.$

$\Rightarrow \exists \varepsilon_X>0$ s.t. $\forall x \in X$ $|z-x| > \varepsilon_X.$

& $\exists \varepsilon_Y>0$ s.t. $\forall y \in Y$ $|z-y| > \varepsilon_Y$

Let $\varepsilon = \min\{\varepsilon_X, \varepsilon_Y\}$

Then for ε $\forall w \in X \cup Y$, $|z-w| > \varepsilon.$

$\Rightarrow z \notin \overline{X \cup Y}$

$\Rightarrow z \notin X \cup Y$

Contradiction $\Rightarrow$ $z \in \overline{X} \cup \overline{Y}$. $\Rightarrow$ $\overline{X \cup Y} \subseteq \overline{X} \cup \overline{Y}$

$\Leftarrow$ Let $z \in \overline{X} \cup \overline{Y}$

If $z \in \overline{X}$ $\Rightarrow \forall \varepsilon > 0 \; \exists \; x \in X \subseteq X \cup Y$
s.t. $|x - z| \leq \varepsilon$.

$\Rightarrow z \in \overline{X \cup Y}$

Similarly $z \in \overline{Y} \Rightarrow z \in \overline{X \cup Y}$

$\Rightarrow$ $\overline{X} \cup \overline{Y} \subseteq \overline{X \cup Y}$

Thus $\overline{X} \cup \overline{Y} = \overline{X \cup Y}$.

③ $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$

Let $z \in \overline{X \cap Y}$

$\Rightarrow \forall \varepsilon > 0 \; \exists \; w \in X \cap Y$ s.t. $|z - w| \leq \varepsilon$.

But $w \in X$ & $w \in Y$ both.

$\Rightarrow$ $w \in \overline{X}$ & $w \in \overline{Y}$

$\Rightarrow$ $\overline{X \cap Y} \subseteq \overline{X} \cap \overline{Y}$.

④ If $X \subseteq Y$ then $\overline{X} \subseteq \overline{Y}$.

Proof: Let $z \in \overline{X}$.

$\forall \varepsilon > 0 \; \exists \; x \in X \subseteq Y$ s.t. $|z - x| \leq \varepsilon$.

$\Rightarrow \forall \varepsilon > 0 \; \exists \; x \in Y$ s.t. $|z - x| \leq \varepsilon$

$\Rightarrow$ $z \in \overline{Y}$

$\Rightarrow$ $\overline{X} \subseteq \overline{Y}$.

Remark: $X = Y \Leftrightarrow X \subseteq Y$ & $Y \subseteq X$.

Lemma: The closure of (a,b), $(a,b]$, $[a,b)$, $[a,b]$

is $[a,b]$.

Proof: $\overline{(a,b)} = [a,b]$

$\Rightarrow$

$[a,b] \subseteq \overline{(a,b)}$

$[a,b] \subseteq \overline{(a,b)}$

Let $x \in [a,b]$

→ If $x \in (a,b)$ then $\forall \varepsilon > 0$ $\exists$ $y = x \in (a,b)$ s.t. $|x-y| = 0 \leq \varepsilon$.

→ If $x = a$, then ~~take~~ $y = a + \varepsilon/2 \in (a,b)$ s.t. $|x-y| = \varepsilon/2 \leq \varepsilon$ $\forall \varepsilon > 0$.

→ If $x = b$, then take $y = b - \varepsilon/2 \in (a,b)$ s.t. $|x-y| = \varepsilon/2 \leq \varepsilon$ $\forall \varepsilon > 0$.

$\Rightarrow x \in \overline{(a,b)}$

Thus $[a,b] \subseteq \overline{(a,b)}$.

$\Leftarrow$ $\overline{(a,b)} \subseteq [a,b]$

Equivalent to showing that no point outside $[a,b]$ belongs to $\overline{(a,b)}$.

<u>Case 1:</u> $x < a$

Let $\varepsilon_0 = a - x > 0$

Then for any $y \in (a,b)$, we have $y > a$

Then $|y-x| = y - x > a - x > 0$.

$\Rightarrow$ For $x < a$ there is no point in (a,b) s.t. $|y-x| \leq \varepsilon_0$.

$\Rightarrow x \notin \overline{(a,b)}$

<u>Case 2:</u> $x > b$

$\varepsilon_0 = x - b$

Then for any $y \in (a,b)$ $|x-y| = x - y > x - b = \varepsilon_0$.

$\Rightarrow x \notin \overline{(a,b)}$

$\Rightarrow x \notin [a,b] \Rightarrow x \notin \overline{(a,b)}$

or $x \in \overline{(a,b)} \Rightarrow x \in [a,b]$

$\Rightarrow \overline{(a,b)} \subseteq [a,b]$

Combining $\overline{(a,b)} = [a,b]$.

Claim: $\overline{\mathbb{N}} = \mathbb{N}$, $\overline{\mathbb{Z}} = \mathbb{Z}$, $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{R}} = \mathbb{R}$.

(1a) $\mathbb{N} \subseteq \overline{\mathbb{N}}$.

Let $x \in \mathbb{N}$. Then take $y = x \in \mathbb{N}$. and $|y-x| = 0 < \varepsilon$.

$\Rightarrow \quad N \subseteq \overline{N}.$

(1b) $\overline{N} \subseteq N$

$x \in \overline{N} \Rightarrow x \in N \iff x \notin N \Rightarrow x \notin \overline{N}.$

Let $x \notin N$

$\delta := \min\{ |x-n| : n \in N \}$

Choose $\varepsilon = \delta/2$

$\Rightarrow \forall y \in N. \qquad |x-y| > \delta > \varepsilon$

$\Rightarrow x \notin \overline{N}$

$\Rightarrow \overline{N} \subseteq N.$

Thus $\overline{N} = N.$

(3a) $\overline{Q} \subseteq R. \Rightarrow x \in \overline{Q} \Rightarrow x \in R$

$\iff x \notin R \Rightarrow x \notin \overline{Q}$

$\delta = \min\{ |x-r| : r \in R \}$

$\Rightarrow \forall r \in R \quad |x-r| > \delta/2 = \varepsilon$

$\Rightarrow \forall r \in Q \quad |x-r| > \varepsilon$

$\Rightarrow x \notin \overline{Q}.$

(3b) $x \in R \Rightarrow x \in \overline{Q}$

If $x \in R \Rightarrow x \in Q$ then ~~take~~ $y = x \in Q$

s.t. $|y-x| < \varepsilon \;\; \forall \;\; \varepsilon > 0.$

$\Rightarrow \overline{Q} = R.$

$\overline{\phi} = \phi :$

No $x \in R$ can satisfy "$\exists y \in \phi$ with $|x-y| < \varepsilon$"

$\Rightarrow \overline{\phi}$ is empty.

§ Lecture 19.1

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Lemma: Let $X \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Then x is an adherent point of X iff $\exists$ a sequence $(a_n)_{n=0}^{\infty}$ with $a_n \in X$, converges to x.

Proof: $\Rightarrow$ Let x be an adherent point of X.

$\forall \varepsilon > 0 \quad \exists y \in X$ s.t. $|x-y| \leq \varepsilon$.

Now construct a sequence $(a_n)_{n \geq 0}$ as follows.

For each $n \in \mathbb{N}$, take $\varepsilon = \frac{1}{n+1}$

Since x is an adherent point of $X \Rightarrow \exists a_n \in X$

s.t. $|x - a_n| \leq \frac{1}{n+1}$

$$\Rightarrow x - \frac{1}{n+1} \leq a_n \leq x + \frac{1}{n+1}$$

$$\Rightarrow \lim_{n \to \infty} a_n = x.$$

$\Leftarrow$ Let $(a_n)_{n \geq 0} \subseteq X$ with $\lim_{n \to \infty} a_n = x$.

$\forall \varepsilon > 0 \quad \exists N$ s.t. $\forall n \geq N$

$|a_n - x| \leq \varepsilon$

Then any $\underbrace{a_n}_{n \geq N} \in X$ is a point s.t. $|x - a_n| \leq \varepsilon$ $\forall \varepsilon > 0$.

Closed subset: $E \subseteq \mathbb{R}$ is said to be closed (in $\mathbb{R}$) if $\overline{E} = E$.

Example: $[a,b]$, $[a,\infty)$, $(-\infty,b]$, $(-\infty,\infty)$ are closed.

(a,b), $(a,b]$, $[a,b)$, (a,∞), $(-\infty,b)$

are open as $\overline{(a,b)} = [a,b]$

$(a,b] = [a,b]$

$[a,b) = [a,b]$

$(a,\infty) = [a,\infty)$

$(-\infty,b) = [-\infty,b]$.

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \phi$ are closed.

$\mathbb{Q}$ is open set.

Limit points: Let $X \subseteq \mathbb{R}$. We say that x is a limit point (cluster point) of X iff it is an adherent point of $X \setminus \{x\}$. We say that x is an isolated point of X if $x \in X$ and $\exists$ $\varepsilon > 0$ s.t. $|x-y| > \varepsilon$ $\forall$ $y \in X \setminus \{x\}$.

Ex: $X = (1,2) \cup \{3\}$

3 is an adherent point of X.

But 3 is not an adherent point of $X - \{3\} = (1,2)$

But 3 is an isolated point of X. Because $3 \in X$ and $\exists$ $\varepsilon = 0.5$ s.t. $|3-y| > 0.5$ $\forall$ $y \in (1,2)$

2 is limit point as 2 is adherent point of $X - \{2\} = X$.

2 is not an isolated point as $2 \notin X$.

Proposition: Let $X \subseteq \mathbb{R}$. Let X' be the set of all limit points. Then

$$\overline{X} = X \cup X'.$$

Proof: $\overline{X}$ = set of all adherent points.

(1) $\Rightarrow$ $X \cup X' \subseteq \overline{X}$

Let $x \in X \cup X'$

(1a) let $x \in X \Rightarrow \forall \varepsilon > 0$ $\exists$ $y = x \in X$ s.t. $|y - x| \leq \varepsilon$.

$\Rightarrow x \in \overline{X}$

(1b) let $x \in X' \Rightarrow \forall \varepsilon > 0$ $\exists$ $y \in X - \{x\}$ s.t. $|y - x| \leq \varepsilon$.

This $y \in X \Rightarrow x \in \overline{X}$

$\Rightarrow \quad X \cup X' \subseteq \overline{X}.$

(2) $\Leftarrow$ Let $x \in \overline{X} \Rightarrow x \in X \cup X'$.

(2a) If $x \in X' \Rightarrow x \in X \cup X' \Rightarrow \quad \overline{X} \subseteq X \cup X'$

(2b) If $x \notin X' \Rightarrow \exists\, \varepsilon_0 > 0$ s.t $\forall\, y \in X - \{x\}$

$|y - x| > \varepsilon_0$

Since $x \in \overline{X}$ we have that $\exists\, y \in X$ s.t.

$|y - x| \leq \varepsilon_0.$

This can happen only when $y = x$

$\Rightarrow x \in X$

$\Rightarrow x \in X \cup X'$

$\overline{X} \subseteq X \cup X'$

or $\boxed{\overline{X} = X \cup X'.}$ Standard definition.

Proposition: x is a limit point of $X \subseteq \mathbb{R}$ iff $\exists$ a sequence $(a_n)_{n \geq 0}$, consisting of elements from $X - \{x\}$, such that $\lim_{n \to \infty} a_n = x$.

Proof: $\Rightarrow$ Let x be a limit point of X.

This implies $\forall\, \varepsilon > 0 \quad \exists\, y \in X - \{x\}$ s.t

$|y - x| \leq \varepsilon.$

Or for $\varepsilon_n = \frac{1}{n+1} \quad \exists\, a_n \in X - \{x\}$ s.t

$|x - a_n| \leq \frac{1}{n+1} \quad \forall\, n \geq 0.$

Indeed such a sequence satisfies

$x - \frac{1}{n+1} \leq a_n \leq x + \frac{1}{n+1}$

$x \leq \lim_{n \to \infty} a_n \leq x$

or $\lim_{n \to \infty} a_n = x.$

$\Leftarrow$ Let $(a_n)_{n \geq 0}$ be a sequence of elements from

$X - \{x\}$ s.t. $\lim_{n \to \infty} a_n = x$.

$\Rightarrow \forall \varepsilon > 0 \quad \exists N_0 \geq 0$ s.t. $\forall n \geq N_0$

$$|a_n - x| \leq \varepsilon$$

In particular choose any $n \geq N_0$ with

$$a_{N_0} \in X - \{x\} \text{ s.t.}$$

$$|a_{N_0} - x| \leq \varepsilon.$$

$\Rightarrow x$ is a limit point of X.

§ Lecture 19.2

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Bounded sets: A set $X \subseteq \mathbb{R}$ is said to be bounded iff $\exists M>0$ s.t. $X \subset [-M, M]$.

Heine-Borel theorem: Let $X \subseteq \mathbb{R}$. Then following statements are equivalents

① X is closed and bounded.

② Given any sequence $(a_n)_{n\geq 0}$ with $a_n \in X$, there exists a subsequence $(a_{n_j})_{j\geq 0}$ that converges to some $L \in X$.

Proof:

① ⇒ ②

Assume X is closed and bounded.

Let $(a_n)_{n\geq 0}$ be a sequence with $a_n \in X$.

Since X is bounded (a_n) is bounded sequence.

From Bolzano-Weierstrass theorem there is a subsequence (a_{n_j}) that converges to $L \in \mathbb{R}$.

Since X is closed and (a_{n_j}) is a sequence of elements of X that converges to L. This implies $L \in \overline{X} = X$.

② ⇒ ①.

Boundedness: Suppose X is unbounded. Then for each n we can pick $x_n \in X$ s.t. $|x_n| > n$.

$\Rightarrow (x_n)$ is unbounded and cannot have any convergent subsequence. This contradicts ②

$\Rightarrow$ X is bounded.

Closeness: Suppose X is not closed.

$\Rightarrow \exists$ a limit point x s.t. $x \notin X$.

Since x is a limit point there exists a sequence

$(b_n) \subset X$ s.t. $b_n \to x$.

By (2) there exists a subsequence (b_{n_j}) converging

to $L \in X$.

But $\lim_{n\to\infty} b_n = \lim_{j\to\infty} b_{n_j} = L = x$.

$\Rightarrow x \in X$. Contradiction

$\Rightarrow$ X is closed.