§ Lecture 18.0

Monday, 13 October 2025 16:53

Q. $\lim_{n\to\infty} n^{1/n} = 1.$

Answer: $\limsup_{n\to\infty} n^{1/n} \le \limsup_{n\to\infty} \frac{n+1}{n} = 1$

$\liminf_{n\to\infty} n^{1/n} \ge \liminf_{n\to\infty} \frac{n+1}{n} = 1$

$\Rightarrow$ We have $1 \le \liminf_{n\to\infty} n^{1/n} \le \limsup_{n\to\infty} n^{1/n} \le 1$

$\Rightarrow \limsup_{n\to\infty} n^{1/n} = 1 = \liminf_{n\to\infty} n^{1/n} = \lim_{n\to\infty} n^{1/n}.$

Raabe's test:

Let $(a_n)_{n\ge m}$ be a sequence of nonzero real numbers.

Define

$$R_n = n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right). \text{ Then}$$

1. If $\liminf_{n\to\infty} R_n > 1$ then $\sum_n a_n$ is absolutely convergent.
2. If $\limsup_{n\to\infty} R_n < 1$ then " is not " ".
3. Inconclusive otherwise.

Proof: Let $L = \liminf_{n\to\infty} R_n > 1.$

$\exists\ \epsilon > 0$ s.t.

$$1 < L-\varepsilon < L.$$

$\Rightarrow \sup(\bar{R}_N)_{N\ge m} = L > L-\varepsilon$

$\Rightarrow \exists\ N_0 \ge m$ s.t. $\bar{R}_{N_0} > L-\varepsilon$

$$\inf(R_n)_{n\ge N_0} > L-\varepsilon$$

$\Rightarrow \exists N_0 \ge m$ s.t. $\forall\ n \ge N_0$

$$R_n > L-\varepsilon$$

$$n\left(1 - \frac{|a_{n+1}|}{|a_n|}\right) > L-\varepsilon$$

or $\frac{|a_{n+1}|}{|a_n|} < 1 - \left(\frac{L-\varepsilon}{n}\right)$

or $|a_{n+1}| < |a_n|\left(1 - \left(\frac{L-\varepsilon}{n}\right)\right) \quad \forall n \ge N_0$

$$|a_{N_0+1}| < |a_{N_0}|\left(1 - \frac{L-\varepsilon}{N_0}\right)$$

$\Rightarrow |a_n| < |a_{N_0}| \prod_{j=N_0}^{n-1}\left(1 - \frac{L-\varepsilon}{k}\right) \quad \forall\ n > N_0$

Note that $(1+x)^{\gamma} \ge 1+\gamma x$ for $\gamma \ge 1$ & $x \ge -1$

$x = -1/k \ge -1 \qquad k \ge N_0 \ge m$

$\gamma = L-\varepsilon > 1$

$\Rightarrow \left(1 - \frac{L-\varepsilon}{k}\right) \le \left(1 - \frac{1}{k}\right)^{L-\varepsilon}$
$= \left(\frac{k-1}{k}\right)^{L-\varepsilon}$

$\Rightarrow |a_n| < |a_{N_0}| \left(\frac{N_0-1}{N_0}\right)^{L-\varepsilon}\left(\frac{N_0}{N_0+1}\right)^{L-\varepsilon}\cdots\left(\frac{n-2}{n-1}\right)^{L-\varepsilon}$

$= |a_{N_0}|\left(\frac{N_0-1}{n-1}\right)^{L-\varepsilon}$

$= |a_{N_0}|(N_0-1)^{L-\varepsilon}\left(\frac{1}{n-1}\right)^{L-\varepsilon}$

↑ But $\sum_{n=N_0}^{\infty}\left(\frac{1}{n-1}\right)^{L-\varepsilon}$ is convergent.

$\Rightarrow \sum_{n=N_0}^{\infty} |a_n|$ is convergent.

$\Rightarrow \sum_{n=m}^{\infty} |a_n|$ is convergent $\Rightarrow \sum_{n=m}^{\infty} a_n$ is absolutely convergent.

(ii) $L = \limsup_{n\to\infty} R_n < 1.$

$\Rightarrow \exists\ \varepsilon > 0.\quad L < L+\varepsilon < 1$

$\inf(R_N^+)_{N\ge m} = L < L+\varepsilon$

$\Rightarrow \exists\ N_0$ s.t. $R_{N_0}^+ < L+\varepsilon$

$\Rightarrow \exists N_0$ s.t. $\forall\ n \ge N_0 \quad R_n < L+\varepsilon$

$$n\left(1 - \left|\frac{a_{n+1}}{a_n}\right|\right) < L+\varepsilon$$

$$|a_{n+1}| > |a_n|\left(1 - \frac{L+\varepsilon}{n}\right)$$

$\Rightarrow |a_n| > |a_{N_0}| \prod_{k=N_0}^{n-1}\left(1 - \frac{L+\varepsilon}{k}\right)$ (Induction)

$\ge |a_{N_0}|\left(1 - \sum_{k=N_0}^{n-1}\frac{L+\varepsilon}{k}\right)$

And $\sum_{k=N_0}^{n-1}\frac{1}{k}$ diverges when $n\to\infty$.

$\Rightarrow |a_n|$ diverges.

EX: Consider the following example

$$\sum_{n=1}^{\infty}\frac{1}{n}\times\frac{(2n-1)!!}{(2n)!!}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\,\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$$

$\frac{a_{n+1}}{a_n} = \frac{n}{n+1}\,\frac{1\cdot 3\cdots(2n+1)}{2\cdot 4\cdots(2n+2)}\,\frac{(2\cdot 4\cdots 2n)}{1\cdot 3\cdots(2n-1)}$

$= \frac{n\times(2n+1)}{2n\times(n+1)}$

$= \frac{n+1/2}{n+1}$

$= \frac{1+1/2n}{1+1/n}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1.$

$1 - \frac{a_{n+1}}{a_n} = \frac{1+\frac{1}{n}-1-1/2n}{1+1/n}$

$= \frac{1/2n}{1+1/n}$

$\lim_{n\to\infty}\left[n\left(1-\frac{a_{n+1}}{a_n}\right)\right] = \frac{1}{2}\lim_{n\to\infty}\frac{1}{1+1/n} = \frac{1}{2}. < 1$

The series is divergent.

EX2: $a_n = \frac{1}{n^q}\,\frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}$

$\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^q\frac{(2n+1)}{(2n+2)}$

$= \left(\frac{1}{1+\frac{1}{n}}\right)^q\left(\frac{2n+1}{2n+2}\right)$

$= \left(1+\frac{1}{n}\right)^{-q}\left(1+\frac{1}{2n}\right)\left(1+\frac{1}{n}\right)^{-1}$

$(a+b)^n = \sum_{m=0}^{n} {}^nC_m\, a^{n-m} b^m$

${}^nC_m = \frac{n!}{m!\,(n-m)!} \qquad n! = n(n-1)\cdots 1.$

$(1+x)^n = \sum_{m=0}^{n} {}^nC_m\, x^m$

$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$

If $|x| < 1$ & $r > 0$

$(1+x)^{-r} = 1 - rx + \frac{-r(-r-1)}{2!}x^2 + \cdots$

$= 1 - rx + \frac{r(r+1)}{2}x^2 + \cdots$

Now $\left(1+\frac{1}{n}\right)^{-q-1}\left(1+\frac{1}{2n}\right)$

$= \left(1 - \frac{(q+1)}{n} + \frac{(q+1)(q+2)}{2n^2} - \cdots\right)\left(1+\frac{1}{2n}\right)$

$= 1 + \left(-\frac{(q+1)}{n} + \frac{1}{2n}\right) + \left(\frac{(q+1)(q+2)}{2n^2} - \frac{(q+1)}{2n^2}\right) + \cdots$

$= 1 + \frac{1-2q-2}{2n} + o\left(\frac{1}{n^2}\right) \quad \cdots$

$n\left(1 - \frac{a_{n+1}}{a_n}\right) = \frac{2q+1}{2} + o\left(\frac{1}{n}\right)$

$\lim_{n\to\infty}\left[n\left(1-\frac{a_{n+1}}{a_n}\right)\right] = \frac{2q+1}{2}$

$\Rightarrow$ From Raabe's test

$\frac{2q+1}{2} > 1 \Leftrightarrow 2q > 1 \Leftrightarrow q > 1/2$ (Convergent)

$q < 1/2$ (Divergent)

$q = 1/2$ Inconclusive.

§ Lecture 18.1

Monday, 13 October 2025 20:59

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\frac{a_{n+1}}{a_n} = \frac{n(n+1)}{(n+1)(n+2)} = \frac{1}{1+2/n}$$

$$1 - \frac{a_{n+1}}{a_n} = 1 - (1+2/n)^{-1}$$

$$= 1 - (1 - 2/n + 4/n^2 - \cdots)$$

$$= 2/n - 4/n^2$$

$$\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) = 2$$

$$S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{N+1}$$

$$\lim_{N\to\infty} S_N = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}. \quad \text{(Question)}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+k)(n+k+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+k} - \frac{1}{n+k+1}\right)$$

$$S_N = \left(\frac{1}{k+1} - \frac{1}{k+2}\right) + \left(\frac{1}{k+2} - \frac{1}{k+3}\right)$$

$$+ \cdots + \left(\frac{1}{N+k} - \frac{1}{N+k+1}\right)$$

$$= \frac{1}{k+1} - \frac{1}{N+k+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 ; \quad \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{(n+k)(n+k+1)(n+k+2)}.$$

$$\Leftrightarrow \quad A(n+k+1)(n+k+2) + B(n+k)(n+k+2) + C(n+k)(n+k+1) = 1$$

$$A + B + C = 0$$

$$(2k+3)A + (2k+2)B + (2k+1)C = 0$$

$$(k+1)(k+2)A + k(k+2)B + (k(k+1))C = 1$$

$$\left(-(2k+3) + 2k+2\right)B + \left(-(2k+3) + (2k+1)\right)C = 0.$$

$$-B - 2C = 0 \Rightarrow B = -2C$$

$$A = -B - C$$

$$= 2C - C = C.$$

$$\left((k+1)(k+2) - 2k(k+2) + k(k+1)\right)C = 1$$

$$(k^2 + 3k + 2 - 2k^2 - 4k + k^2 + k)C = 1$$

$$C = 1/2.$$

$$B = -1$$

$$A = 1/2$$

$$\frac{1}{2(n+k)} - \frac{1}{(n+k+1)} + \frac{1}{2(n+k+2)}$$

$$S_N = \frac{1}{2}\sum_{n=1}^{N} \frac{1}{(n+k)} - \sum_{n=1}^{N} \frac{1}{(n+k+1)} + \frac{1}{2}\sum_{n=1}^{N} \frac{1}{(n+k+2)}.$$

$$= \frac{1}{2}\sum_{n=1}^{N}\left(\frac{1}{n+k}\right) + \left(-\sum_{n=1}^{N}\frac{1}{n+k} + \left(\frac{1}{k+1} - \frac{1}{N+k+1}\right)\right)$$

$$+ \frac{1}{2}\left(\sum_{n=1}^{N}\frac{1}{(n+k)} - \frac{1}{k+1} - \frac{1}{k+2} + \frac{1}{N+k+1} + \frac{1}{N+k}\right)$$

$$= \left(\frac{1}{k+1} - \frac{1}{N+k+1}\right) - \frac{1}{2(k+1)} - \frac{1}{2(k+2)} + \frac{1}{2(N+k+1)} + \frac{1}{2(N+k)}$$

$$= \frac{1}{2(k+1)} - \frac{1}{2(k+2)} - \frac{1}{2(N+k+1)} + \frac{1}{2(N+k)}$$

$$= \frac{1}{2(k+1)(k+2)}.$$

$k=0$ $\quad \frac{1}{4}$

$k=1$ $\quad \frac{1}{12}$

$k=2$ $\quad \frac{1}{24}.$

Rearrangement of series:

Proposition:

Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series.

Let $f: \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=0}^{\infty} a_{f(n)}$ is also absolutely convergent and

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{f(n)}.$$

Remark: If a series contains only positive terms and is convergent, then it is absolutely convergent.

Consider: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

This is a convergent series from alternating series test as $\lim_{n\to\infty} \frac{1}{n} = 0$.

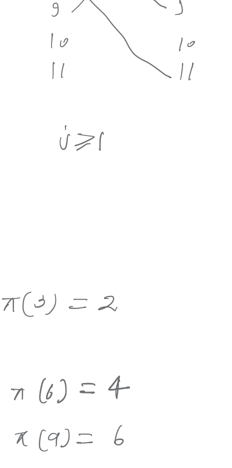
But it is not an absolutely convergent series.

Let $$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3}\right) - \frac{1}{4} + \cdots$$

$$\left[\begin{array}{l} \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\ \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \end{array}\right]$$

Now let us rearrange:



Let $a_n = \frac{(-1)^{n+1}}{n}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Define $\pi: \mathbb{N} \to \mathbb{N}$ by

$$\pi(3j-2) = 4j-3$$

$$\pi(3j-1) = 4j-1 \qquad j \geq 1$$

$$\pi(3j) = 2j$$

$\pi(1) = 1, \quad \pi(2) = 3, \quad \pi(3) = 2$

$\pi(4) = 5, \quad \pi(5) = 7, \quad \pi(6) = 4$

$\pi(7) = 9, \quad \pi(8) = 11, \quad \pi(9) = 6$

$$\sum_{n=1}^{\infty} a_{\pi(n)} = a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + \cdots$$

$$= \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right)$$

$$+ \left(\frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \cdots$$

Partial sum after N blocks:

$$S_{3N} = \sum_{j=1}^{N} \left(\frac{1}{4j-3} + \frac{1}{4j-1} - \frac{1}{2j}\right) \quad \text{(First 3N terms)}$$

$$\sum_{j=1}^{N} \left(\frac{1}{4j-3} + \frac{1}{4j-1}\right) = \left(1 + \frac{1}{3}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{9} + \frac{1}{11}\right) + \cdots$$

$$+ \frac{1}{4N-3} + \frac{1}{4N-1}$$

$$= \sum_{j=1}^{4N} \frac{1}{j} - \frac{1}{2}\sum_{j=1}^{2N} \frac{1}{j}$$

$$= H_{4N} - H_{2N}/2$$

$$\Rightarrow \quad S_{3N} = H_{4N} - \frac{1}{2}H_{2N} - \frac{1}{2}H_N$$

Use $H_N \approx \ln N + \gamma$

$$S_{3N} = \ln 4N + \gamma - \frac{1}{2}\ln 2N - \frac{1}{2}\gamma - \frac{\ln N}{2} - \frac{1}{2}\gamma$$

$$= 2\ln 2 - \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2$$

$$\lim_{N\to\infty} S_{3N} = \frac{3}{2}\ln 2.$$