§ Lecture 17.0

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Cauchy criterion: Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with $a_n \geq 0$ and $a_{n+1} \leq a_n$ $\forall\, n \geq 1$. Then $\sum_{n=1}^{\infty} a_n$ is convergent iff $\sum_{k=0}^{\infty} 2^k a_{2^k}$ is convergent.

Ex. $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent when $q>1$ and divergent when $q \leq 1$. q is a rational number.

Proof: $a_n = \frac{1}{n^q} \geq 0 \quad \forall q$

$$a_{n+1} = \frac{1}{(n+1)^q} \leq \frac{1}{n^q} = a_n \quad \forall q$$

From Cauchy criterion $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent iff

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^q} \quad \text{is convergent.}$$

or

$$\sum_{k=0}^{\infty} 2^{(1-q)k} \quad \text{is convergent.}$$

This is a geometric series and is convergent iff

$|2^{1-q}| < 1$ and divergent otherwise.

$\downarrow$

This is always positive.

$$\Rightarrow \quad 2^{1-q} < 1$$

$$2^{1-q} < 2^0 \iff 1-q < 0 \Rightarrow q > 1.$$

$$\zeta(q) = \sum_{n=1}^{\infty} \frac{1}{n^q} \;;$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent eventhough $\lim_{n\to\infty} a_n = 0$.

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Rearrangement of series:

Proposition:

Let $\sum_{n=0}^{\infty} a_n$ be an absolutely convergent series.

Let $f: \mathbb{N} \to \mathbb{N}$ be a bijection. Then $\sum_{n=0}^{\infty} a_{f(n)}$ is also absolutely convergent and

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} a_{f(n)}$$

Remark: If a series contains only positive terms and is convergent, then it is absolutely convergent.

Consider: $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

This is a convergent series from alternating series test as $\lim_{n\to\infty} \frac{1}{n} = 0$.

But it is not an absolutely convergent series.

Let
$$S = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$$
$$= \left(1 - \frac{1}{2} + \frac{1}{3}\right) - \frac{1}{4} + \cdots$$

$$\left[\begin{array}{l} \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \\ \ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \cdots \end{array}\right]$$

Now let us rearrange:

Let $a_n = \frac{(-1)^{n+1}}{n}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}.$$

Define $\pi: \mathbb{N} \to \mathbb{N}$ by

$$\pi(3j-2) = 4j-3$$
$$\pi(3j-1) = 4j-1 \qquad j \geq 1$$
$$\pi(3j) = 2j$$

$\pi(1) = 1, \quad \pi(2) = 3, \quad \pi(3) = 2$

$\pi(4) = 5, \quad \pi(5) = 7, \quad \pi(6) = 4$

$\pi(7) = 9, \quad \pi(8) = 11, \quad \pi(9) = 6$

$$\sum_{n=1}^{\infty} a_{\pi(n)} = a_1 + a_3 + a_2 + a_5 + a_7 + a_4 + \cdots$$
$$= \left(1 + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \left(\frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \cdots$$

Partial sum after N blocks:

$$S_{3N} = \sum_{j=1}^{N} \left(\frac{1}{4j-3} + \frac{1}{4j-1} - \frac{1}{2j}\right) \quad \text{(First } 3N \text{ terms)}$$

$$\sum_{j=1}^{N} \left(\frac{1}{4j-3} + \frac{1}{4j-1}\right) = \left(1+\frac{1}{3}\right) + \left(\frac{1}{5} + \frac{1}{7}\right) + \left(\frac{1}{9} + \frac{1}{11}\right) + \cdots + \frac{1}{4N-3} + \frac{1}{4N-1}$$

$$= \sum_{j=1}^{4N} \frac{1}{j} - \frac{1}{2} \sum_{j=1}^{2N} \frac{1}{j}$$

$$= H_{4N} - H_{2N}/2$$

$$\Rightarrow \quad S_{3N} = H_{4N} - \frac{1}{2} H_{2N} - \frac{1}{2} H_N$$

Use $H_N \approx \ln N + \gamma$

$$S_{3N} = \ln 4N + \gamma - \frac{1}{2}\ln 2N - \frac{1}{2}\gamma - \frac{\ln N}{2} - \frac{1}{2}\gamma$$

$$= 2\ln 2 - \frac{1}{2}\ln 2 = \frac{3}{2}\ln 2$$

$$\lim_{N\to\infty} S_{3N} = \frac{3}{2}\ln 2.$$

Recall, associativity rule:

$$(a+b)+c = a+(b+c)$$

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \cdots$$
$$= (a_1 + a_2) + a_3 + (a_4 + a_5) + a_6 + \cdots$$
$$\neq a_1 + (a_2 + a_3) + a_4 + (a_5 + a_6) + \cdots$$

Infact, one can prove that for not absolutely convergent series

$$\sum_{n=1}^{\infty} a_n = L \quad \text{for any real number } L.$$

$$\Updownarrow$$

$\exists$ a bijection that will get you any real number as an answer.

§ Lecture 17.1

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The root and ratio test:

Theorem 1: Let $\sum_{n=m}^{\infty} a_n$ be an infinite series of real numbers and

$$\alpha = \limsup_{n\to\infty} |a_n|^{1/n}.$$

Then

$$\sum_{n=m}^{\infty} a_n = \begin{cases} \text{Absolutely convergent} & \text{if } \alpha<1 \\ \text{Not convergent} & \text{if } \alpha>1 \\ \text{Inconclusive} & \text{if } \alpha=1. \end{cases}$$

Proof: Let $b_n := |a_n|^{1/n}$

$$\alpha = \limsup_{n\to\infty} b_n$$
$$= \inf (b_N^+)_{N\geq m} \qquad b_N^+ = \sup(a_n)_{n\geq N}$$

Case 1 $\alpha<1$.

Since $b_n \geq 0 \Rightarrow \alpha \geq 0$

or $0 \leq \alpha < 1$

Then there exists $\varepsilon>0$ s.t. (e.g. $\varepsilon = \frac{1-\alpha}{2}$)

$$0 < \alpha+\varepsilon < 1$$

$$\inf (b_N^+)_{N\geq m} = \alpha < \alpha+\varepsilon$$

$\Rightarrow \exists N_0 \geq m$ s.t. $b_{N_0}^+ < \alpha+\varepsilon$ otherwise $\alpha+\varepsilon$ will be infimum of $(b_N^+)_{N\geq m}$.

We have $\sup (b_n)_{n\geq N_0} < \alpha+\varepsilon$

$\Rightarrow \forall n\geq N_0 \quad b_n < \alpha+\varepsilon$

$\Rightarrow \quad b_n = |a_n|^{1/n} < \alpha+\varepsilon \quad \forall n\geq N_0$

or $\quad |a_n| < (\alpha+\varepsilon)^n \quad \forall n \geq N_0$

$\Rightarrow$ Convergence of $\sum_{n=N_0}^{\infty} (\alpha+\varepsilon)^n$ implies convergence of $\sum_{n=N_0}^{\infty} |a_n|$. [comparison test]

Convergence of $\sum_{n=N_0}^{\infty} (\alpha+\varepsilon)^n \Rightarrow |\alpha+\varepsilon| < 1$
$\Leftrightarrow \alpha < 1-\varepsilon$

$\Rightarrow \alpha<1$ implies absolute convergence of $\sum_{n=N_0}^{\infty} a_n$ or $\sum_{n=m}^{\infty} a_n$ as $\sum_{n=m}^{N_0-1} a_n$ is finite.

Case 2: $\alpha>1$.

$$\limsup_{n\to\infty} b_n = \alpha > 1$$

$$\inf (b_N^+)_{N\geq m} > 1$$

$\forall N \quad b_N^+ > 1$

$\sup (b_n)_{n\geq N} > 1$

If $b_n \leq 1 \quad \forall n\geq N \Rightarrow 1$ is supremum.

$\Rightarrow \exists n_0 \geq N$ s.t. $b_{n_0} > 1$

$\forall N \quad \exists n_0 \geq N \quad b_{n_0} > 1$

$|a_n| > 1$

$\Rightarrow (a_n)_{n\geq N}$ is not convergent to zero.

From zero test $\sum_{n=m}^{\infty} a_n$ is not convergent and hence not absolutely convergent.

Case 3 $\alpha=1$ Consider series

$$\sum_{n=2}^{\infty} \frac{1}{n^2}$$

$$\alpha = \limsup_{n\to\infty} (n^{-2/n})$$

Define sequence

$$y_n = n^{1/n}$$

If $y_n = 1+\delta_n \Rightarrow n = (1+\delta_n)^n \qquad \delta_n>0$

$$n \geq 1 + \frac{n(n-1)}{2}\delta_n^2$$

$\Rightarrow (n-1) \geq \frac{n(n-1)}{2}\delta_n^2$

$\Rightarrow \delta_n^2 \leq 2/n$

$\Rightarrow 0 < \delta_n < \sqrt{2/n}$

$0 < y_n - 1 < \sqrt{2/n}$

$|y_n - 1| < \sqrt{2/n} \qquad \sqrt{2/n} = \varepsilon \Rightarrow n = 2/\varepsilon^2$

$\Rightarrow \forall \varepsilon>0 \quad \exists N = 2/\varepsilon^2$ s.t. $\forall n \geq N$

$$|y_n - 1| \leq \sqrt{\tfrac{2}{N}} = \varepsilon.$$

$\Rightarrow \lim_{n\to\infty} y_n = 1 \Rightarrow \lim_{n\to\infty} y_n^2 = 1$

$\Rightarrow \lim_{n\to\infty} n^{2/n} = 1.$

Thus $\alpha=1$ for $\sum_{n=2}^{\infty} \frac{1}{n^2}$ which is convergent.

$\alpha=1$ But $\sum_{n=2}^{\infty} \frac{1}{n}$ is divergent.

EX: $\sum_{n=1}^{\infty} a_n$, where $a_n = \begin{cases} \left(\frac{1}{2}\right)^n & n=\text{even} \\ \left(\frac{1}{3}\right)^n & n=\text{odd}. \end{cases}$

$$|a_n|^{1/n} = \begin{cases} 1/2 & n \text{ even} \\ 1/3 & n \text{ odd}. \end{cases}$$

$$|a_n|^{1/n} = (1/2, 1/3, 1/2, \ldots)$$

$\Rightarrow \lim_{n\to\infty} |a_n|^{1/n}$ doesn't exist.

But $\limsup_{n\to\infty} |a_n|^{1/n} = \inf(1/2, 1/2, \ldots)$
$= 1/2 < 1.$

$\Rightarrow$ convergent series.

Usually it is difficult to use root test.

Lemma: Let $(c_n)_{n=m}^{\infty}$ be a sequence of positive numbers.

Then

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

Proof: Let $L = \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$

If $L=\infty$ then there is nothing to prove.

$L \neq -\infty$

$\Rightarrow L$ is finite and $L \geq 0$.

Let $\varepsilon>0$. Then $\limsup_{n\to\infty} \frac{c_{n+1}}{c_n} < L+\varepsilon$

$\Rightarrow \inf \left(\frac{c_{n+1}}{c_n}\right)^+ < L+\varepsilon$

$\Rightarrow \exists N \geq m$ s.t. $\left(\frac{c_{N+1}}{c_N}\right)^+ < L+\varepsilon$

$\Rightarrow \frac{c_{n+1}}{c_n} < L+\varepsilon \quad \forall n\geq N$

$c_{N+1} < (L+\varepsilon) c_N$

$c_{N+2} < (L+\varepsilon)^2 c_N$

$c_{N+n-N} < (L+\varepsilon)^{n-N} c_N$

or $\quad c_n < (L+\varepsilon)^{n-N} c_N \qquad n \geq N.$

Let $A = (L+\varepsilon)^{-N} c_N$

$\Rightarrow c_n < A(L+\varepsilon)^n$

$\Rightarrow c_n^{1/n} < A^{1/n}(L+\varepsilon)$

$$\limsup_{n\to\infty} c_n^{1/n} \leq \limsup_{n\to\infty} (A^{1/n})(L+\varepsilon)$$
$$= L+\varepsilon \qquad \forall \varepsilon>0$$

$$\Rightarrow \boxed{\limsup_{n\to\infty} c_n^{1/n} \leq L}$$

Similarly

$$L = \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$$

Let $\varepsilon>0$. Then

$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} > L-\varepsilon$$

$\Rightarrow \exists N$ s.t. $\forall n \geq N$

$$\frac{c_{n+1}}{c_n} > L-\varepsilon$$

$c_{n+1} > c_n(L-\varepsilon) \qquad \forall n \geq N$

or $\quad c_n > c_N (L-\varepsilon)^n (L-\varepsilon)^{-N}$

$\Rightarrow c_n^{1/n} > B^{1/n}(L-\varepsilon), \quad B = c_N(L-\varepsilon)^{-N}$

$\Rightarrow \liminf_{n\to\infty} (c_n^{1/n}) \geq L-\varepsilon \quad \forall \varepsilon>0$

$\Rightarrow \liminf_{n\to\infty} (c_n)^{1/n} \geq \liminf_{n\to\infty} \frac{c_{n+1}}{c_n}$

Ratio test: Let $\sum_{n=m}^{\infty} a_n$ be a series of nonzero numbers.

Then

1. If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ Then $\sum_{n=m}^{\infty} a_n$ is absolutely convergent.
2. If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ Then $\sum_{n=m}^{\infty} a_n$ is not convergent.
3. In remaining cases, no conclusion.

§ Lecture 17.2

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Claim: $\lim_{n\to\infty} n^{1/n} = 1.$

Proof: $\limsup_{n\to\infty} n^{1/n} \le \limsup_{n\to\infty} \frac{n+1}{n} = 1$

$\liminf_{n\to\infty} n^{1/n} \ge \liminf_{n\to\infty} \frac{n+1}{n} = 1$

$\Rightarrow \lim_{n\to\infty} n^{1/n} = 1.$

Raabe's test:

Let $\sum_{n=m}^{\infty} a_n$ be the series. and $a_n > 0$. Define

$$R_n = n\left(1 - \frac{a_{n+1}}{a_n}\right)$$

① If $\liminf_{n\to\infty} R_n > 1 \Rightarrow \sum_{n=m}^{\infty} a_n$ is converges.

② If $\limsup_{n\to\infty} R_n < 1 \Rightarrow$ " diverges.

③ inconclusive otherwise.

Proof: Let $L = \liminf_{n\to\infty} R_n$

Case 1: $L > 1$ $\quad 1 < L - \varepsilon < L$

$$\liminf_{n\to\infty} R_n = \sup (R_N^+)_{N\ge m} = L > L - \varepsilon$$

$\Rightarrow \exists N_0$ s.t. $R^+_{N_0} > L - \varepsilon$

$\forall n \ge N_0 \quad R_n > L - \varepsilon$

$\Rightarrow \quad L - \varepsilon < R_n = n\left(1 - \frac{a_{n+1}}{a_n}\right) \quad \forall n \ge N_0$

$\Rightarrow \quad \frac{a_{n+1}}{a_n} \le 1 - \left(\frac{L-\varepsilon}{n}\right)$

$a_{n+1} \le a_n \left(1 - \left(\frac{L-\varepsilon}{n}\right)\right)$

$a_n \le a_N \prod_{k=N_0}^{n-1} \left(1 - \frac{L-\varepsilon}{k}\right) \quad n \ge N_0.$

Use $\left(1 - \frac{1}{k}\right)^{L-\varepsilon} \ge 1 - \frac{L-\varepsilon}{k}$

$\Rightarrow \quad a_n \le a_N \prod_{k=N_0}^{n-1} \left(1 - \frac{1}{k}\right)^{L-\varepsilon}$

$= a_N \left(\frac{N_0 - 1}{n-1}\right)^{L-\varepsilon}$