§ Lecture 16.0

Monday, 6 October 2025 19:41

Absolute convergence: $\sum_{n=m}^{\infty} a_n$ is called absolutely convergent iff $\sum_{n=m}^{\infty} |a_n|$ is convergent.

Lemma: If a series is absolutely convergent then it is convergent. and

conditionally convergent!

$$\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|.$$

Proof: Let $s_k = \sum_{n=m}^{k} a_n$

$$t_k = \sum_{n=m}^{k} |a_n|$$

Note that for $p+1 \leq q$

$$|s_q - s_p| = \left| \sum_{n=m}^{q} a_n - \sum_{n=m}^{p} a_n \right|$$

$$= \left| \sum_{n=p+1}^{q} a_n \right|$$

$$\leq \sum_{n=p+1}^{q} |a_n|$$

$$= t_q - t_p$$

Because $(t_k)_{k \geq m}$ converges $\Rightarrow$ (t_k) is a Cauchy sequence. So $\forall \varepsilon > 0$, $\exists N \geq m$ s.t.

if $q, p \geq N$ then

$$|t_q - t_p| \leq \varepsilon.$$

$$\Rightarrow |s_q - s_p| \leq \varepsilon$$

$\Rightarrow$ (s_k) is a Cauchy sequence. $\Rightarrow$ (s_k) is convergent.

Since $|s_k| \leq t_k \quad \forall k \geq m$

$$\Rightarrow \lim_{k \to \infty} |s_k| \leq \lim_{k \to \infty} t_k$$

or $\left| \sum_{n=m}^{\infty} a_n \right| \leq \sum_{n=m}^{\infty} |a_n|$

[Converse is not true].

§ Lecture 16.1

Thursday, 2 October 2025 22:32

Alternating series test : Let $\sum_{n=m}^{\infty} a_n$ be an infinite series with $a_n \geq 0$ and $a_{n+1} \leq a_n$ $\forall n \geq m$.

Then $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent iff $(a_n)_{n\geq m}$ converges to zero.

Proof: $\Rightarrow$ From zero test if $\sum_{n=m}^{\infty} (-1)^n a_n$ is convergent

then $\lim_{n\to\infty} [(-1)^n a_n] = 0$

or $\forall \varepsilon > 0$ $\exists N \geq m$ s.t $\forall$ $n \geq m$

$$|(-1)^n a_n - 0| \leq \varepsilon$$

$$\Rightarrow |a_n - 0| \leq \varepsilon$$

$\Rightarrow$ $(a_n)_{n\geq m}$ converges to zero.

$\Leftarrow$ Suppose conversely that (a_n) converges to 0.

Let $S_N = \sum_{n=m}^{N} (-1)^n a_n$ $\forall n \geq m$.

Our goal is to show that $(S_N)_N$ converges.

Note that m is arbitrary.

Define $E_k = S_{m+2k} = \sum_{n=m}^{m+2k} (-1)^n a_n$ $k \geq 0$

$O_k = S_{m+2k+1} = \sum_{n=m}^{m+2k+1} (-1)^n a_n$ $k \geq 0$

Then $(S_N)_{N\geq m} = (S_m, S_{m+1}, S_{m+2}, S_{m+3}, \cdots)$

$$= (E_0, O_0, E_1, O_1, E_2, O_2, \cdots)$$

$$E_k - E_{k+1} = \sum_{n=m}^{m+2k} (-1)^n a_n - \sum_{n=m}^{m+2k+2} (-1)^n a_n$$

$$= (-1)^{m+2k} [a_{m+2k+1} - a_{m+2k+2}]$$

$$= (-1)^m \underbrace{[a_{m+2k+1} - a_{m+2k+2}]}_{\geq 0}$$

$$O_k - O_{k+1} = (-1)^{m+2k+3} [a_{m+2k+2} - a_{m+2k+3}]$$

$$= (-1)^{m+1} \underbrace{[a_{m+2k+2} - a_{m+2k+3}]}_{\geq 0}$$

If m is even. then	m is odd
$E_k \geq E_{k+1}$	$E_k \leq E_{k+1}$
$O_k \leq O_{k+1}$	$O_k \geq O_{k+1}$.

Suppose m is odd.

$(S_N)_{N\geq m} = (E_0, O_0, E_1, O_1, E_2, O_2, \cdots)$

$(E_k)_{k\geq 0}$ is increasing sequence

$(O_k)_{k\geq 0}$ is decreasing sequence.

Note that $E_k - O_k = \sum_{n=m}^{m+2k} a_n (-1)^n - \sum_{n=m}^{m+2k+1} (-1)^n a_n$

$$= -(-1)^{m+2k+1} a_{m+2k+1}$$

$$= (-1)^m a_{m+2k+1}$$

$$= -a_{m+2k+1}$$

$$\leq 0$$

$\Rightarrow$ $E_k \leq O_k$ $\forall k \geq 0$

$\Rightarrow$ (E_k) is an increasing sequence and upper bounded by any O_ℓ. $(\ell \leq k)$

$\Rightarrow$ (O_k) is decreasing sequence and lower bounded by an earlier E_ℓ. $(\ell \leq k)$

$\boxed{E_0 \leq E_1 \cdots \leq E_k \leq O_k \leq O_{k-1} \cdots \leq O_0}$

$\Rightarrow$ Both (E_k) and (O_k) converges.

let $L_{even} = \lim_{k\to\infty} E_k$

$L_{odd} = \lim_{k\to\infty} O_k$

Then

$E_k - O_k = -a_{m+2k+1}$

$L_{even} - L_{odd} = -\lim_{k\to\infty} a_{m+2k+1}$

$= 0$

Call $L = L_{even} = L_{odd}$.

Given $\varepsilon > 0$ choose K large enough s.t. $\forall k \geq K$

$|E_k - L| \leq \varepsilon/2$

$|O_k - L| \leq \varepsilon/2$

Then $M \geq K$ s.t $\forall$ $p, q \geq M$

$$|S_p - S_q| \leq |S_p - L| + |S_q - L|$$

$$\leq \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon$$

$\Rightarrow$ $(S_N)_{N\geq m}$ is a Cauchy sequence and hence convergent.

Note that $E_0 \leq \cdots \leq E_k \leq O_k \leq O_{k-1} \leq \cdots \leq O_0$

$\Rightarrow$ E_0 and O_0 are global lower and upper bounds respectively. (m even).

§ Lecture 16.2

Monday, 6 October 2025 20:26

$\left(\frac{1}{n}\right)_{n\geq 1}$ is a convergent sequence converging to zero.

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent series.

But we can't say yet whether $\sum_{n=1}^{\infty} \frac{1}{n}$ is convergent.

Series laws:

Let $\sum_{n=m}^{\infty} a_n = x$, $\sum_{n=m}^{\infty} b_n = y$

Then $\sum_{n=m}^{\infty} (a_n + b_n) = x+y$

Proof: $S_N = \sum_{n=m}^{N} a_n$

$\tilde{S}_N = \sum_{n=m}^{N} b_n$

$S'_N = \sum_{n=m}^{N} (a_n + b_n) = S_N + \tilde{S}_N$

Given $\lim_{N\to\infty} S_N = x$

$\lim_{N\to\infty} \tilde{S}_N = y$

$\Rightarrow \lim_{N\to\infty} (S_N + \tilde{S}_N) = \lim_{N\to\infty} S_N + \lim_{N\to\infty} \tilde{S}_N$

$= x+y$

(b) $\sum_{n=m}^{\infty} (c\, a_n) = cx$ if $\sum_{n=m}^{\infty} a_n = x$.

Again from limit law of sequences.

(c) Let $\sum_{n=m}^{\infty} a_n$ be a series of real numbers, and let $k\geq 0$ be an integer. If one of the two series $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m+k}^{\infty} a_n$ are convergent then the other one is also and

$$\sum_{n=m}^{\infty} a_n = \sum_{n=m}^{m+k-1} a_n + \sum_{n=m+k}^{\infty} a_n$$

Proof: let $S_N = \sum_{n=m}^{N} a_n$

$T_N = \sum_{n=m+k}^{N} a_n$

For $N \geq m+k$

$S_N = \sum_{n=m}^{m+k-1} a_n + T_N$

If $T_N \to T \Rightarrow S_N \to F+T$

$S_N = F + T_N$

or $T_n = F - S_N$ if $S_N \to S \Rightarrow T_N \to F-S$.

Comparison test:

Let $\sum_{n=m}^{\infty} a_n$ and $\sum_{n=m}^{\infty} b_n$ be two series and

$|a_n| \leq b_n \quad \forall\, n \geq m$.

Then if $\sum_n b_n$ is convergent then $\sum_n a_n$ is absolutely convergent.

$\sum_{n=m}^{\infty} |a_n| \leq \sum_{n=m}^{\infty} b_n$.

Proof: Let $S_N = \sum_{n=m}^{N} |a_n|$

$T_N = \sum_{n=m}^{N} b_n$

Then $S_N \leq T_N$

But (T_N) is convergent $\Rightarrow (T_N)$ is bounded.

$\Rightarrow \exists\, M$ s.t. $|T_N| \leq M \quad \forall\, N \geq m$

Note that $T_N \leq |T_N| \leq M$

$\Rightarrow S_N \leq M$

(S_N) is an increasing sequence that is bounded from above $\Rightarrow (S_N)$ is convergent.

$\Rightarrow \sum_{n=m}^{\infty} |a_n|$ is convergent $\Rightarrow \sum_{n=m}^{\infty} a_n$ is absolutely convergent.

§ Lecture 16.3

Monday, 6 October 2025 21:49

Cauchy Criterion: Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence s.t $a_n \geq 0$ & $a_{n+1} \leq a_n \ \forall n \geq m$.

Then $\sum_{n=1}^{\infty} a_n$ is convergent iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k} \text{ is convergent.}$$

Proof: $S_N = \sum_{n=1}^{N} a_n$

$$T_K = \sum_{k=0}^{K} 2^k a_{2^k}.$$

If we prove that $(S_N)_{N=1}^{\infty}$ is bounded iff $(T_K)_{K=0}^{\infty}$ is bounded. Then ??

$$S_{2^{K+1}-1} \leq T_K \leq 2 S_{2^K}$$

Proof: For $K=0$

$$S_{2^{K+1}-1} = S_1 = a_1, \quad T_K = T_0 = a_1, \quad 2S_{2^K} = 2S_1 = 2a_1$$

$$a_1 \leq a_1 \leq 2a_1$$

Suppose $S_{2^{K+1}-1} \leq T_K \leq 2S_{2^K}$ is true.

Then $$T_{K+1} = \sum_{k=0}^{K+1} 2^k a_{2^k}$$

$$= T_K + 2^{K+1} a_{2^{K+1}}$$

$$S_{2^{K+1}} = \sum_{n=0}^{2^{K+1}} a_n$$

$$= \sum_{n=0}^{2^K} a_n + \sum_{n=2^K+1}^{2^{K+1}} a_n$$

$a_{n+1} \leq a_n$

$$\geq S_{2^K} + a_{2^{K+1}} \left(2^{K+1} - 2^K\right)$$

$$= S_{2^K} + 2^K a_{2^{K+1}}$$

$$\Rightarrow \boxed{2 S_{2^{K+1}} \geq 2 S_{2^K} + 2^{K+1} a_{2^{K+1}}} \longrightarrow 2$$

$$S_{2^{K+2}-1} = \sum_{k=0}^{2^{K+2}-1} a_n$$

$$= \sum_{k=0}^{2^{K+1}-1} a_n + \sum_{k=2^{K+1}}^{2^{K+2}-1} a_n$$

$$\leq S_{2^{K+1}-1} + a_{2^{K+1}} \left(2^{K+2} - 2^{K+1}\right)$$

$$= S_{2^{K+1}-1} + 2^{K+1} a_{2^{K+1}}$$

$$S_{2^{K+2}-1} \leq S_{2^{K+1}-1} + 2^{K+1} a_{2^{K+1}}$$

$$\leq T_K + 2^{K+1} a_{2^{K+1}}$$

$$\leq 2S_{2^K} + 2^{K+1} a_{2^{K+1}}$$

$$\leq 2 S_{2^{K+1}}$$

$$\Rightarrow S_{2^{K+2}-1} \leq T_{K+1} \leq 2 S_{2^{K+1}}$$

Thus $$\boxed{S_{2^{K+1}-1} \leq T_K \leq 2 S_{2^K}} \quad K \geq 0.$$

If $(S_N)_{N \geq 1}$ is bounded then $(S_{2^K})_K$ is bounded.

$\Rightarrow (T_K)$ is bounded

If T_K is bounded then $\exists M$ s.t. $S_{2^{K+1}-1} \leq M \ \forall K \geq 0.$

But $2^{K+1} - 1 \geq K+1$

$\Rightarrow S_{K+1} \leq M \quad \forall K \geq 0$

$\Rightarrow (S_N)_{N=1}^{\infty}$ is bounded

Example: Let $q > 0$ be a rational number.

Then $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent when $q > 1$ and divergent when $q \leq 1$.

Proof: Sequence $\left(\frac{1}{n^q}\right)_{n=1}^{\infty}$ is non negative and decreasing for any $q \geq 0$.

From Cauchy criterion $\sum_{n=1}^{\infty} \frac{1}{n^q}$ is convergent iff

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^q} \text{ is convergent.}$$

$$\sum_{k=0}^{\infty} \left(2^{1-q}\right)^k.$$

This is geometric series.

convergent iff $|2^{1-q}| < 1$

divergent when $|2^{1-q}| \geq 1$

$$-1 < 2^{1-q} < 1$$

$$-2^0 < 2^{1-q} < 2^0$$

$$1-q < 0 \Rightarrow q > 1$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. while $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Rearrangement of series:

If $\sum_{n=0}^{\infty} a_n$ is absolutely convergent then $\sum_{n=0}^{\infty} a_{f(n)}$ is also absolutely convergent for any bijection $f: \mathbb{N} \to \mathbb{N}$.

Proof: