§ Lecture 13.1

Thursday, 25 September 2025 15:17

Subsequence:

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be sequences of real numbers. We say that (b_n) is a subsequence of $(a_n)_{n=0}^{\infty}$ iff $\exists$ a $\underbrace{\text{strictly increasing}}_{f(n+1)>f(n)}$ function $f: \mathbb{N} \to \mathbb{N}$ s.t.

$$b_n = a_{f(n)} \quad \forall n \in \mathbb{N}$$

Remark: f is injective. (Because it is strictly increasing)

(It may not be bijective)

$(a_0\ a_2\ a_4 \cdots)$ is a subsequence of $(a_0\ a_1 \cdots)$. $(f(n) = 2n)$

$1\ -1\ 1\ -1 \cdots$

$$\left.\begin{aligned} (b_n) &= (1\ 1\ 1) = (a_{2n}) \\ (c_n) &= (-1\ -1 \cdots) = (a_{2n+1}) \end{aligned}\right\}\text{subsequences.}$$

"Order preserving subset".

$((-1)^n)_{n\geq 0}$ is not convergent but subsequences $((-1)^{2n})_{n\geq 0}$ and $((-1)^{2n+1})_{n\geq 0}$ are convergent.

Proposition:

Let $(a_n)_{n\geq 0}$ be a sequence of real numbers and let $L \in \mathbb{R}$. The following statements are equivalent.

(a) The sequence converges to L.

(b) Every subsequence of $(a_n)_{n\geq 0}$ converges to L.

(a) $\Leftrightarrow$ (b).

Proof: (a) $\Rightarrow$ (b)

Suppose $(a_n)_{n\geq 0}$ converges to L.

$\Rightarrow \exists N \geq 0$ s.t. $\forall n \geq N$

$$|a_n - L| \leq \varepsilon \quad \forall \varepsilon > 0.$$

Define an arbitrary sequence

$$b_n = a_{f(n)} \quad (f \text{ is strictly increasing } f^n)$$

Since $f(n)$ is strictly increasing $\exists M$ s.t.

$\forall n \geq M \Rightarrow f(n) \geq N$ (for any fixed N)

$\Rightarrow \exists M$ s.t. $\forall n \geq M$

$$|b_n - L| = |a_{f(n)} - L| \leq \varepsilon \quad \forall \varepsilon > 0.$$

$\Rightarrow (b_n)_{n\geq 0}$ converges to L.

But (b_n) is an arbitrary subsequence.

This completes the proof.

(b) $\Rightarrow$ (a)

$\neg$(a) $\Rightarrow \neg$(b)

The sequence $(a_n)_{n\geq 0}$ doesn't converge to L.

$\Rightarrow \exists \varepsilon_0 > 0$ s.t. $\forall N \geq 0 \ \exists n \geq N$ with

$$|a_n - L| > \varepsilon_0. \quad \to (1)$$

Now let us construct the following subsequence.

Let us construct an increasing sequence of indices $(n_k)_{k=0}^{\infty}$ inductively so that

$$|a_{n_k} - L| > \varepsilon_0.$$

Step 1: For $k=0$ apply eq^n (1) with $N=0$.

This implies there exists $n_0 \geq 0$ with

$$|a_{n_0} - L| > \varepsilon_0$$

Step 2: Suppose n_k is chosen with $|a_{n_k} - L| > \varepsilon_0$.

Apply eq^n (1) for $N = n_k + 1$. Then there exists $n_{k+1} \geq n_k + 1$ s.t. $|a_{n_{k+1}} - L| > \varepsilon_0$.

Thus we have a subsequence

$(a_{n_0}\ a_{n_1} \cdots)$ or $(a_{n_k})_{k=0}^{\infty}$

s.t. $\exists \varepsilon_0$ s.t. $\forall k \geq 0 \ \exists n_k \geq 0$ with

$$|a_{n_k} - L| > \varepsilon_0$$

$\Rightarrow (a_{n_k})_{k\geq 0}$ is convergent to L.

This completes the proof.

Proposition:

(a) L is a limit point of $(a_n)_{n\geq 0}$.

(b) There exists a subsequence of $(a_n)_{n\geq 0}$ which converges to L.

Proof: (a) $\Rightarrow$ (b)

Assume L is a limit point of $(a_n)_{n\geq 0}$.

$\Rightarrow \forall \varepsilon > 0 \ \forall N \geq 0 \ \exists n \geq N$ with

$$|a_n - L| \leq \varepsilon. \quad \to (1)$$

Define a sequence $(\varepsilon_k)_{k\geq 0}$, where $\varepsilon_k = 1/(k+1)$

Choose indices $n_0 < n_1 \cdots$ s.t.

$$|a_{n_k} - L| \leq \varepsilon_k = \frac{1}{k+1} \quad \forall k \geq 0$$

How?

(1) Apply eq^n (1) for $\varepsilon = 1$, $N = 0$

This gives some $n_0 \geq 0$ s.t. $|a_{n_0} - L| \leq 1$.

(2) Assume n_k is constructed s.t.

$$|a_{n_k} - L| \leq \varepsilon_k.$$

Apply (1) for $N = n_k + 1$, $\varepsilon_{k+1} = 1/(k+2)$

$\exists n_{k+1} \geq n_k + 1$ s.t.

$$|a_{n_{k+1}} - L| \leq \varepsilon_{k+1}$$

So now we have a subsequence

$(a_{n_k})_{k\geq 0}$ s.t.

$$|a_{n_k} - L| \leq \frac{1}{k+1} \quad \forall k \geq 0.$$

For any given $\varepsilon > 0$ choose K large enough s.t. $1/(K+1) \leq \varepsilon$. Then for every $k \geq K$

$$|a_{n_k} - L| \leq \frac{1}{k+1} \leq \frac{1}{K+1} \leq \varepsilon$$

$\Rightarrow (a_{n_k})_{k\geq 0}$ converges to L.

(b) $\Rightarrow$ (a). Let a subsequence $(a_{n_k})_{k\geq 0}$ converges to L.

Fix $\varepsilon > 0$ and any $N \in \mathbb{N}$.

Since $n_k \to \infty$ as $k \to \infty$, there is a $\tilde{K}$ with $n_{\tilde{K}} \geq N$.

Because (a_{n_k}) converges to L, $\exists K'$ s.t. $\forall k \geq K'$

$$|a_{n_k} - L| \leq \varepsilon.$$

Take $k \geq \max\{\tilde{K}, K'\} \Rightarrow k \geq \tilde{K}$
$\Rightarrow n_k \geq N$

$$\Rightarrow |a_{n_k} - L| \leq \varepsilon.$$

$\Rightarrow \forall N \ \& \ \forall \varepsilon > 0 \ \exists n \geq N$ s.t.

$$|a_n - L| \leq \varepsilon$$

$\Rightarrow L$ is a limit point of $(a_n)_{n\geq 0}$.

§ Lecture 13.2

Thursday, 25 September 2025 21:12

Bolzano–Weierstrass theorem:

Let $(a_n)_{n\geq 0}$ be a bounded sequence, i.e. $\exists\, M>0$ s.t. $|a_n| \leq M$ $\forall n \in \mathbb{N}$. Then there is at least one subsequence of $(a_n)_{n\geq 0}$ which converges.

Proof: Let $L = \limsup_{n\to\infty} a_n$

Note that $-M \leq a_n \leq M$ $\forall n \geq 0$.

Taking $\limsup_{n\to\infty} (-M) \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} M$

$\Leftrightarrow$ $-M \leq L \leq M$

$\Rightarrow$ L is finite. $\Rightarrow$ L is a limit point of $(a_n)_{n\geq 0}$.

$\Rightarrow$ There exists a subsequence that converges.

Real exponentiation:

Let $x>0$ and α be a real number. Let $(q_n)_{n\geq 1}$ be any sequence of rationals converging to α. Then $(x^{q_n})_{n\geq 1}$ converges. Moreover if $(q_n')_{n\geq 1}$ another sequence converging to α. Then

$$\lim_{n\to\infty} x^{q_n} = \lim_{n\to\infty} x^{q_n'} \underbrace{:= x^{\alpha}}_{\text{Definition.}}$$

Proof: $x=1$ trivial.

Case: $x>1$ | $x<1$ similar.

$\Downarrow$

It is sufficient to prove that $(x^{q_n})_{n\geq 1}$ is a Cauchy sequence.

Let $q_n \geq q_m \Rightarrow x^{q_n} \geq x^{q_m}$

$$|x^{q_n} - x^{q_m}| = x^{q_n} - x^{q_m} = x^{q_m}(x^{q_n - q_m} - 1)$$

Since (q_n) is a convergent sequence, it is bounded.

Let $q_n \leq M$ $\forall n \geq 1$.

$x^{q_n} \leq x^M$

Then

$$\Rightarrow |x^{q_m} - x^{q_n}| \leq x^M (x^{q_n - q_m} - 1)$$

Note that the sequence $(x^{1/k})_{k\geq 1}$ is convergent to 1.

$\Rightarrow \exists\, k\geq 1$ s.t. $|x^{1/k} - 1| \leq \varepsilon \times x^{-M}$ for some $\varepsilon > 0$.

Since (q_n) is a Cauchy sequence $\exists\, N\geq 1$ s.t. $\forall$ $n,m \geq N$

$$q_n - q_m \leq 1/k$$

$$\Rightarrow |x^{q_m} - x^{q_n}| \leq x^M (x^{1/k} - 1) \leq \varepsilon$$

Similarly $q_n \leq q_m$.

$\Rightarrow$ $\forall \varepsilon > 0$ $\exists$ N s.t. $\forall$ $n,m \geq N$

$$|x^{q_m} - x^{q_n}| \leq \varepsilon$$

$\Rightarrow$ $(x^{q_n})_{n\geq 1}$ is a Cauchy sequence and hence convergent.

Claim 2: $\lim_{n\to\infty} x^{q_n - q_n'} = 1$.

Proof: Write $q_n - q_n' = r_n$

$\Rightarrow$ $(r_n)_{n\geq 1}$ converges to 0.

$\Rightarrow$ $-1/k \leq r_n \leq 1/k$ for all $k>0$. $k \geq 1$.

$$\underset{\uparrow 1}{x^{-1/k}} \leq x^{r_n} \leq \underset{\uparrow 1}{x^{1/k}}$$

$\Rightarrow$ (x^{r_n}) converges to 1.

§ Lecture 14.0

Monday, 29 September 2025 20:53

Finite series:

Let m, n be integers and let $(a_i)_{i=m}^{n}$ be a finite sequence of real numbers. Then finite series $\sum_{i=m}^{n} a_i$ is defined recursively as

$$\sum_{i=m}^{n} a_i := 0 \quad \text{if } n < m$$

If $\sum_{i=m}^{n} a_i$ is defined then

$$\sum_{i=m}^{n+1} a_i = \sum_{i=m}^{n} a_i + a_{n+1} \qquad n \geq m-1$$

Thus $\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$

Further $\sum_{i=m}^{n} a_i = \sum_{j=m}^{n} a_j$ (i, j are called dummy indices)

Properties:

① $\sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}$

Proof: Since we can change dummy index.

Define $j = i+k \Rightarrow i = j-k$

when $i = m \Rightarrow j = m+k$

$i = n \Rightarrow j = n+k$

$$\Rightarrow \sum_{i=m}^{n} a_i = \sum_{j=m+k}^{n+k} a_{j-k}$$

② $\sum_{i=m}^{n} a_i + \sum_{j=n+1}^{p} a_j = \sum_{l=m}^{p} a_l$

Proof: fix m & n $(m \leq n)$. Prove by induction on $p \geq n+1$.

Base case: $p = n+1$

$$L.H.S = \sum_{i=m}^{n} a_i + \sum_{j=n+1}^{n+1} a_j = \sum_{i=m}^{n} a_i + a_{n+1}$$

$$= \sum_{i=m}^{n+1} a_i$$

$$= R.H.S.$$

Assume $p = q \geq n+1$ is true, i.e.

$$\sum_{i=m}^{n} a_i + \sum_{j=n+1}^{q} a_j = \sum_{l=m}^{q} a_l$$

Then

$$\sum_{i=m}^{n} a_i + \sum_{j=n+1}^{q+1} a_j$$

$$= \sum_{i=m}^{n} a_i + \sum_{j=n+1}^{q} a_j + a_{q+1}$$

$$= \sum_{l=1}^{q} a_l + a_{q+1}$$

$$= \sum_{l=1}^{q+1} a_l$$

$$\Rightarrow \sum_{i=m}^{n} a_i + \sum_{j=n+1}^{p} a_j = \sum_{l=1}^{p} a_l \text{ is true for all } p \geq n+1.$$

$\longrightarrow$ If $p < n+1$ then (Trivial) $\sum_{j=n+1}^{p} a_j = 0$.

③ $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{j=m}^{n} b_j$

Proof: If $n < m$ (Trivial)

$0 = 0$.

Let $n \geq m$.

Base case: $n = m$ then

$$\sum_{i=m}^{m} (a_i + b_i) = a_m + b_m$$

Let $\sum_{i=m}^{n} (a_i + b_i) = \sum_{i=m}^{n} a_i + \sum_{j=m}^{n} b_j$ is true

Then

$$\sum_{i=m}^{n+1} (a_i + b_i) = \sum_{i=m}^{n} (a_i + b_i) + a_{n+1} + b_{n+1}$$

$$= \left(\sum_{i=m}^{n} a_i + \sum_{j=m}^{n} b_j\right) + a_{n+1} + b_{n+1}$$

$$= \sum_{i=m}^{n+1} a_i + \sum_{j=m}^{n+1} b_j$$

④ $\sum_{i=m}^{n} (c a_i) = c \sum_{j=m}^{n} a_j$

Proof: $n < m$ trivial.

$n = m$

$L.H.S = c a_m = R.H.S.$

Let $\sum_{i=m}^{n} (c a_i) = c \sum_{j=m}^{n} a_j$ is true

$$\sum_{i=m}^{n+1} (c a_i) = \sum_{i=m}^{n} c a_i + c a_{n+1}$$

$$= c\left(\sum_{i=m}^{n} a_i + a_{n+1}\right)$$

$$= c \sum_{i=m}^{n+1} a_i.$$

⑤ $\left|\sum_{i=m}^{n} a_i\right| \leq \sum_{i=m}^{n} |a_i|$.

⑥ If $a_i \leq b_i \ \forall \ m \leq i \leq n$.

$$\sum_{i=m}^{n} a_i \leq \sum_{i=m}^{n} b_i.$$

Again induction!!

§ Lecture 14.1

Monday, 29 September 2025 21:23

Summation over finite sets:

Let X be a finite set of n elements and let $f: X \to \mathbb{R}$ be a function. Then $\sum_{x \in X} f(x)$ is defined as follows.

① select any bijection $g: \{i \in \mathbb{N} : 1 \le i \le n\} \to X$

(This exists by definition).

② $\sum_{x \in X} f(x) = \sum_{i=1}^{n} f(g(i))$

e.g. $X = \{a, b, c\}$ $n = 3$

Define $g(1) = a$ $g(2) = b$, $g(3) = c$

Then $\sum_{x \in X} f(x) = \sum_{i=1}^{3} f(g(i))$

$= f(a) + f(b) + f(c)$.

If we define another bijection h s.t

$h(1) = c$, $h(2) = a$, $h(3) = b$

$\Rightarrow \sum_{x \in X} f(x) = f(c) + f(a) + f(b) = \sum_{i=1}^{3} f(g(i))$

$= \sum_{i=1}^{3} f(h(i))$.

Proposition:

Let X be a finite set with n element. and let $f: X \to \mathbb{R}$ be a function. Let $g, h : \{i \in \mathbb{N} : 1 \le i \le n\} \to X$ be two bijections. Then

$$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f(h(i))$$

Proof: Let $P(n)$ be the following property:

" For any set X of n elements and function $f: X \to \mathbb{R}$ and any two bijections g, h from $\{i \in \mathbb{N} : 1 \le i \le n\}$ to $\mathbb{R}$

$\sum_{i=1}^{n} f(g(i)) = \sum_{i=1}^{n} f\,h(i)$. "

Base case: $n = 0$

$P(0)$ is true because

$\sum_{i=1}^{0} f(g(i)) = 0 = \sum_{i=1}^{0} f(h(i))$

Assume $P(n)$ is true.

Then for $P(n+1)$, set X has $(n+1)$ elements.

$\sum_{i=1}^{n+1} f(g(i)) = \sum_{i=1}^{n} f(g(i)) + f(g(n+1))$

Since g is a bijection $g(n+1) = x \in X$.

$\Rightarrow \sum_{i=1}^{n+1} f(g(i)) = \sum_{i=1}^{n} f(g(i)) + f(x)$

If $h(n+1) = x$ we are done.

If not then there exists some index j s.t.

$h(j) = x$.

Then $\sum_{i=1}^{n+1} f(h(i)) = \sum_{i=1}^{j-1} f(h(i)) + f(x) + \sum_{i=j+1}^{n+1} f(h(i))$

$= \sum_{i=1}^{j-1} f(h(i)) + f(x) + \sum_{i=j}^{n} f(h(i+1))$

Define $\tilde{h} : \{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$

$$\tilde{h}(i) = \begin{cases} h(i) & \text{for } i < j \\ h(i+1) & \text{for } i \ge j \end{cases}$$

RHS $= \sum_{i=1}^{j-1} f(\tilde{h}(i)) + \sum_{i=j}^{n} f(\tilde{h}(i)) + f(x)$

$= \sum_{i=1}^{n} f(\tilde{h}(i)) + f(x)$

Recall $g(n+1) = x$

Then g restricted to $\{i \in \mathbb{N} : 1 \le i \le n\}$ is a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\} \to X - \{x\}$.

The function $\tilde{h}$ is also a bijection.

$\Rightarrow \sum_{i=1}^{n} f(\tilde{h}(i)) = \sum_{i=1}^{n} f(g(i))$

$\Rightarrow \sum_{i=1}^{n+1} f(h(i)) = \sum_{i=1}^{n} f(g(i)) + f(x)$

$= \sum_{i=1}^{n+1} f(g(i))$.

Properties: ① $\sum_{x \in \phi} f(x) = 0$ (The only bijection is the empty map)

$\sum_{i=1}^{0} f(g(i)) = 0$.

② Let X be a finite set and $f: X \to \mathbb{R}$ be a function. and $g: Y \to X$ be a bijection then

$\sum_{x \in X} f(x) = \sum_{y \in Y} f(g(y))$

Proof: Let $h: \{i \in \mathbb{N} : 1 \le i \le n\} \to X$

$u: \{i \in \mathbb{N} : 1 \le i \le n\} \to Y$

be bijections then. $g \circ u$ is a bijection from $\{i \in \mathbb{N} : 1 \le i \le n\} \to X$.

$\Rightarrow \sum_{x \in X} f(x) = \sum_{i=1}^{n} f(g \circ u(i))$

$= \sum_{i=1}^{n} (f \circ g)(u(i))$

$= \sum_{y \in Y} f \circ g(y)$

$= \sum_{y \in Y} f(g(y))$.

③ Let X, Y be finite sets with $X \cap Y = \phi$ then

$\sum_{x \in X \cup Y} f(x) = \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)$

$f: X \cup Y \to \mathbb{R}$.

Proof: Let X, Y have n, m elements, respectively.

n, m could be zero too.

$g: \{1, 2, .. n\} \to X$

$h: \{1, 2, .. m\} \to Y$

Define a map

$k: \{1, 2, \dots, m+n\} \to X \cup Y$ by

$$k(i) = \begin{cases} g(i) & 1 \le i \le n \\ h(i-n) & n+1 \le i \le m+n \end{cases}$$

Because $X \cap Y = \phi$ k is a bijection onto $X \cup Y$.

$\sum_{z \in X \cup Y} f(z) = \sum_{i=1}^{m+n} f(k(i))$

$= \sum_{i=1}^{n} f(k(i)) + \sum_{i=n+1}^{n+m} f(k(i))$

$= \sum_{i=1}^{n} f(g(i)) + \sum_{i=n+1}^{n+m} f(h(i-n))$

$j = i - n$

$= \sum_{x \in X} f(x) + \sum_{j=1}^{m} f(h(j))$

$= \sum_{x \in X} f(x) + \sum_{y \in Y} f(y)$.

Lemma: Let X, Y be finite sets. Let $f: X \times Y \to \mathbb{R}$ be a function then

$$\sum_{x \in X} \left(\sum_{y \in Y} f(x, y) \right) = \sum_{(x,y) \in X \times Y} f(x, y).$$

Proof: Let n be the number of elements in X.

We will use induction on n.

$P(n)$: Let lemma is true for any set X with n elements any finite set Y and any function $f: X \times Y \to \mathbb{R}$.

We aim to prove that $P(n)$ is true for all n.

Base case: $P(0)$, ie. X is empty

$\sum_{x \in \phi} \left(\sum_{y \in Y} f(x, y) \right) = 0$

$\phi \times Y = \phi$ why?

$\Rightarrow \sum_{(x,y) \in \phi} f(x, y) = 0$.

Assume $P(n)$ is true.

We want to prove $P(n+1)$.

Let X has $(n+1)$ elements.

Let $x_0 \in X$ then $\underbrace{X - \{x_0\}}_{= X'}$ has n elements.

$X' \cap \{x_0\} = \phi$

$X' \cup \{x_0\} = X$

$\sum_{x \in X} \left(\sum_{y \in Y} f(x, y) \right) = \sum_{x \in X'} \left(\sum_{y \in Y} f(x, y) \right) + \sum_{x \in \{x_0\}} \left(\sum_{y} f(x, y) \right)$

$= \sum_{x \in X'} \left(\sum_{y \in Y} f(x, y) \right) + \sum_{y \in Y} f(x_0, y)$

$= \sum_{(x,y) \in (X' \times Y)} f(x, y) + \sum_{y \in Y} f(x_0, y)$

$= \sum_{(x,y) \in X' \times Y} f(x, y) + \sum_{(x,y) \in \{x_0\} \times Y} f(x, y)$

$(X' \times Y) \cap \{x_0\} \times Y = \phi$

$\Rightarrow = \sum_{(x,y) \in (X' \times Y) \cup (\{x_0\} \times Y)} f(x, y)$

$= \sum_{(x,y) \in X \times Y} f(x, y)$