26/9/25
8:30am

Claim: Let c be a limit point of sequence $(a_n)_{n\geq m}$ of real numbers, then

$$L^- \leq c \leq L^+, \text{ where}$$

$$L^- = \lim_{n\to\infty} \inf a_n \qquad L^+ = \lim_{n\to\infty} \sup a_n$$

Proof:

Since c is a limit point of $(a_n)_{n\geq m}$, we have

$\forall \varepsilon > 0 \quad \forall N \geq m \quad \exists n_0 \geq N$ s.t.

$$|a_{n_0} - c| \leq \varepsilon \longrightarrow (1)$$

Informally ($a_{n_0} = c$ limiting sense)

$$a_N^+ = \sup(a_n)_{n\geq m}$$
$$\geq a_{n_0}$$

$$L^+ = \lim_{N\to\infty}(a_N^+) \geq \lim_{n_0\to\infty} a_{n_0} = c \Rightarrow c < L^+ \longrightarrow (2)$$

$$a_N^- = \inf(a_n)_{n\geq m} \leq a_{n_0}$$

$$\boxed{\lim_{N\to\infty} a_N^- \leq \lim_{n_0\to\infty} a_{n_0}}$$

$\Rightarrow L^- \leq c \longrightarrow$ (2)

From (2) and (3), $L^- \leq c \leq L^+$.

Claim 2: If L^+ is finite, then L^+ is a limit point of $(a_n)_{n \geq m}$.

Proof:

$L^+ = \lim_{k_0 \to \infty} a^+_{k_0}$; $a^+_{k_0} = \sup(a_n)_{n \geq k_0}$

Since L^+ is finite $\Rightarrow (a^+_{k_0})_{k_0 \geq m}$ converges to L^+.

$\Rightarrow \forall \varepsilon > 0 \ \exists N \geq m$ s.t. $\forall \tilde{k} \geq N$

$$|a^+_{\tilde{k}} - L^+| \leq \varepsilon/2 \longrightarrow (2)$$

$\boxed{\tilde{k} = k \text{ 'tilda'}}$

[Recall defn of L^+: L^+ is a limit point of $(a_n)_{n \geq m}$ iff $\forall \varepsilon > 0$, $\forall K \geq m$ $\exists n_0 \geq K$ s.t. $|a_{n_0} - L^+| \leq \varepsilon$] $\rightarrow$ (1)

★★ same as $|a_{n_0} - L^+| < \varepsilon$ ★★

Let us fix some $\varepsilon > 0$ and ~~define~~ fix any $M \geq m$

$$K = \max\{M, N\}$$

Notice that ε and M are arbitrary nos

$\Rightarrow K \geq M$ and $K \geq N$.

From (2),

$$|a^+_K - L^+| \leq \varepsilon \iff \underbrace{L^+ - \varepsilon/2 \leq a^+_K}_{\rightarrow (3)} \leq L^+ + \frac{\varepsilon}{2}$$

$a_k^+ = \sup(a_n)_{n\geq k}$

Q: Can $a_k^+ - \varepsilon/2$ be an upper bound on $(a_n)_{n \geq k}$?

$\Rightarrow$ No!! Otherwise, $a_k^+ - \varepsilon/2$ should be the supremum for $\varepsilon > 0$, not a_k^+!

$\Rightarrow \exists$ some $n_0 \geq k$ s.t.

$$\boxed{a_k^+ - \varepsilon/2 < a_{n_0}} \longrightarrow (4)$$

From (3),

$$L^+ - \varepsilon/2 \leq a_k^+$$

$$\Rightarrow \underbrace{L^+ - \varepsilon \leq a_k^+}_{(3)} \underbrace{- \varepsilon/2 < a_{n_0}}_{(4)} \leq \sup(a_n)_{n\geq k}$$

$$= a_k^+$$

$$\leq L^+ + \varepsilon/2 \quad [\text{From } (3)]$$

$$< L^+ + \varepsilon$$

$$\Rightarrow L^+ - \varepsilon < a_{n_0} < L^+ + \varepsilon$$

(or) $|a_{n_0} - L^+| < \varepsilon$

$\Rightarrow$ For all $\varepsilon > 0$ and $\forall M \geq m \ \exists \ n_0 \geq k \geq M$ s.t. $|a_{n_0} - L^+| < \varepsilon \Rightarrow L^+$ is a limit point of $(a_n)_{n \geq m}$.

LEMMA (Squeeze Test):

Let $(a_n)_{n \geq m}$ and $(b_n)_{n \geq m}$ and $(c_n)_{n \geq m}$ be sequences of real numbers, ~~Then~~ and

$$a_n \leq b_n \leq c_n \quad \forall\ n \geq m.$$

Then if $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\lim_{n\to\infty} b_n = L$.

Proof:

We have, $a_n \leq b_n \leq c_n$

$\Rightarrow \lim_{n\to\infty} \inf(a_n) \leq \lim_{n\to\infty} \inf(b_n) \leq \lim_{n\to\infty} \inf(c_n)$

[We proved it previously]

$\Rightarrow L \leq \lim_{n\to\infty} \inf(b_n) \leq L \Rightarrow \lim_{n\to\infty} \inf(b_n) = L \rightarrow (1)$

Similarly,

$\lim_{n\to\infty} \sup(a_n) \leq \lim_{n\to\infty} \sup(b_n) \leq \lim_{n\to\infty} \sup(c_n)$

$\Rightarrow L \leq \lim_{n\to\infty} \sup(b_n) \leq L \Rightarrow \lim_{n\to\infty} \sup(b_n) = L \rightarrow (2)$

From (1) and (2), $\lim_{n\to\infty} b_n = L$

Hence proved.

Qn: $(a_n)_{n \geq 1}$, where $a_n = 2^{-n}$.

$\lim_{n\to\infty} 2^{-n} = ?$ Ans: 0

Proof ans:

Note that $2^{-n} \leq \frac{1}{n}$ [can be proved by induction]

$$\Rightarrow -\frac{1}{n} \leq 2^{-n} \leq \frac{1}{n}$$

Let $a_n = -1/n$, $b_n = 2^{-n}$, $c_n = 1/n$

WKT $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = 0$

By squeeze theorem, $\lim\limits_{n\to\infty} b_n$ exists and is equal to zero.

$$\lim_{n\to\infty} b_n = 0$$

Subsequences

Let $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ be two sequences of real numbers. We say that $(b_n)_{n\geq 0}$ is a subsequence of $(a_n)_{n\geq 0}$ iff $\exists$ a strictly increasing function $f: \mathbb{N} \to \mathbb{N}$ s.t.

$$b_n = a_{f(n)}$$

(Eg) $a_n = (1\ 2\ 3\ 4\ 5 \ldots)$ $\quad b_n = (2\ 3\ 1)$ ✗

$c_n = (2\ 4\ 6)$ ✓

$\hookrightarrow f(1) = 2 \quad f(2) = 4 \quad f(3) = 6$

$f(1) = 2 \Rightarrow a_{f(1)} = b_1 = a_2 = 2$

$f(2) = 3 \quad a_{f(2)} = b_2 = a_3 = 3$

$f(3) = 1 \quad a_{f(3)} = b_3 = a_1 = 1$

Not an increasing function

PROPOSITION : A sequence $(a_n)_{n\geq 0}$ (infinite) of real numbers converges to L iff all the subsequences (infinite subsequences) of $(a_n)_{n\geq 0}$ converges to L.

PROOF: <u>Proving $\Rightarrow$</u>

Let $(a_n)_{n\geq 0}$ converge to L.

This means that $\forall \varepsilon > 0 \ \exists N_0 \geq 0$ s.t.

$$\forall n \geq N_0 \quad |a_n - L| \leq \varepsilon. \longrightarrow (1)$$

We try to define a subsequence of $(a_n)_{n\geq 0}$ as follows:

$$b_n = a_{f(n)} \quad \left(\begin{array}{l} f \text{ is an arbitrary} \\ \text{increasing function} \end{array}\right)$$

Since $f(n)$ is increasing, $\exists M$ s.t.

$$n \geq M \Rightarrow f(n) \geq N_0 \quad (\text{For fixed } N_0, \text{ from eqn. } (1))$$

From (1)

$\Rightarrow \quad |a_{f(n)} - L| \leq \varepsilon$

(or) $|b_n - L| \leq \varepsilon$

$\Rightarrow (b_n)_{n\geq 0}$ converges to L.

$\forall \varepsilon > 0 \ \exists M$ s.t.

$\forall n \geq M$

$|a_{f(n)} - L| \leq \varepsilon$

Since f is an arbitrary increasing function, every subsequence of (a_n) converges to L //

<u>Proving $\Leftarrow$</u> Claim: If every subsequence of $(a_n)_{n\geq 0}$ converges to L, then $(a_n)_{n\geq 0}$ converges to L.

$\Updownarrow$ CONTRAPOSITIVE

If $(a_n)_{n\geq 0}$ does not converge to L, then $\exists$ some subsequence that doesn't converge to L.

$$\underbrace{\forall \varepsilon > 0 \ \exists N \ \text{s.t.}\ n \geq N,}_{\text{premise}}$$

~~$\forall |a_n - L| > \varepsilon$~~

$$\underbrace{|a_n - L| \leq \varepsilon}_{\text{consequence}}$$

NEGATION:

$$\underbrace{\exists \varepsilon > 0 \ \forall N \ \cancel{\text{s.t.}} \ \exists n \geq N}_{\text{Don't change direction of inequality anywhere in the premise, just interchange } \exists \text{ and } \forall} \ \text{s.t.}\ \underbrace{|a_n - L| > \varepsilon}_{\text{Just reverse the inequality of consequence}}$$

Let such ε be equal to ε_0.

$\forall N \ \exists n \geq N$ s.t. $|a_n - L| > \varepsilon_0 \to (1)$

(i) For $N = 0$ let $n_0 \geq 0$ be an index s.t.

$|a_{n_0} - L| > \varepsilon_0$ [From (1)]

(ii) Assume that $n_k > n_{k-1} \dots > n_0$ exists such that

$|a_{n_k} - L| > \varepsilon_0$

Then take $N = n_k + 1$. Then from (1) $\exists n_{k+1} \geq n_k + 1 > n_k$ s.t. $|a_{n_{k+1}} - L| > \varepsilon_0$

So from induction we found

$$n_0 < n_1 < \cdots < n_k < n_{k+1}$$

s.t.

$$|a_{n_k} - L| > \varepsilon_0 \quad \forall\, k = 0, \ldots$$

For sequence

$$(a_{n_0}\ a_{n_1}\ a_{n_2} \ldots),$$

there is an ε_0 s.t. $\forall\, k \geq 0\ \exists\, n_k$ s.t.

$$|a_{n_k} - L| > \varepsilon_0$$

$\Rightarrow$ We have found a subsequence of (a_n) that doesn't converge to L.

<u>Lemma</u>: Let $(a_n)_{n\geq 0}$ be a sequence of real numbers. Then L is a limit point of $(a_n)_{n\geq 0}$ iff there exists a subsequence of (a_n) that converges to L.