§ Lecture 12.0

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Limit superior and limit inferior:

$$a_n = (-1)^n + \frac{1}{n} \qquad n \geq 1$$

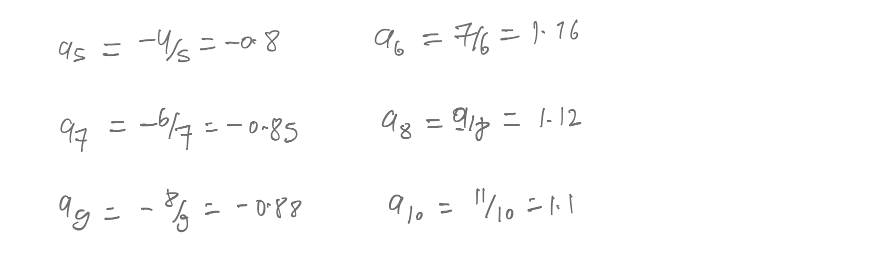
$a_1 = 0$ $\qquad$ $a_2 = 3/2 = 1.5$

$a_3 = -2/3 = -0.66$ $\qquad$ $a_4 = 5/4 = 1.25$

$a_5 = -4/5 = -0.8$ $\qquad$ $a_6 = 7/6 = 1.16$

$a_7 = -6/7 = -0.85$ $\qquad$ $a_8 = 9/8 = 1.12$

$a_9 = -8/9 = -0.88$ $\qquad$ $a_{10} = 11/10 = 1.1$



Claim. Let $(a_n)_{n \geq m}$ be a sequence of real numbers. Let

$$L^+ = \limsup_{n \to \infty} (a_n) \in \mathbb{R} \cup \{\pm\infty\}$$

$$L^- = \liminf_{n \to \infty} (a_n) \in \mathbb{R} \cup \{\pm\infty\}.$$

① For every $x > L^+$, $\exists N \geq m$ s.t.

$x > a_n \quad \forall n \geq N.$

② For every $y < L^-$ $\exists N \geq m$ s.t.

$y < a_n \quad \forall n \geq N.$

Proof: Given $x > L^+$

$\Rightarrow x > \inf (a_N^+)_{N \geq m}$

Since $x > \inf (a_N^+)_{N \geq m}$, there must be at least one N_0 s.t. $x > a_{N_0}^+$

If not then $x \leq a_{N_0}^+ \quad \forall N_0 \geq m$

$\Rightarrow x$ is a lower bound on $(a_N^+)_{N \geq m}$ that is greater than $\inf (a_N^+)_{N \geq m} \Rightarrow x$ is infimum.

Now $x > a_{N_0}^+$
$= \sup (a_n)_{n=N_0}^{\infty}$

$\Rightarrow x > a_n \quad \forall n \geq N_0.$

② $y < L^- = \sup (a_N^-)_{N \geq m}$

$\Rightarrow \exists N_0 \geq m$ s.t. $y < a_{N_0}^-$

otherwise $y \geq a_N^- \quad \forall N \geq m$

$\Rightarrow y$ is a supremum

That is $\exists N_0$ $y < a_{N_0}^-$
$= \inf (a_n)_{n \geq N_0}$

$\Rightarrow y < a_n \quad \forall n \geq N_0.$

③ $\inf (a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup (a_n)_{n=m}^{\infty}$

Proof:

$\inf (a_n)_{n=m}^{\infty} = a_m^- \leq a_{m+1}^- \cdots$

For all $N \geq m$

$\Rightarrow \inf (a_n)_{n=m}^{\infty} \leq a_N^-$
$= \inf (a_n)_{n \geq N}$
$\leq a_N \quad \forall N \geq m$
$\leq \sup (a_n)_{n \geq N} = a_N^+$
$\leq \sup (a_n)_{n=m}^{\infty}$

$\Rightarrow$
$\inf (a_n)_{n \geq m} \leq a_N^- \leq a_N^+ \leq \sup (a_n)_{n=m}^{\infty}$

Taking $\lim_{N \to \infty} \Rightarrow$

$\inf (a_n)_{n \geq m} \leq L^- \leq L^+ \leq \sup (a_n)_{n=m}^{\infty}.$

④ If c is any limit point of $(a_n)_{n \geq m}$

then $L^- \leq c \leq L^+.$

Proof: $\forall N \geq m \quad \exists n_0 \geq N$ s.t.

$|a_{n_0} - c| \leq \varepsilon \quad \forall \varepsilon > 0.$

or $c - \varepsilon \leq a_{n_0} \leq c + \varepsilon \quad \forall \varepsilon > 0.$

$\Rightarrow c \leq a_{n_0} \leq c$

Now $a_N^+ = \sup (a_n)_{n \geq N}$
$\geq a_{n_0} \geq c$

$\lim_{N \to \infty} a_N^+ \geq \lim_{N \to \infty} c$

$\Rightarrow L^+ \geq c.$

$a_N^- = \inf (a_n)_{n \geq N}$
$\leq a_{n_0} \leq c$

$\Rightarrow L^- \leq c$

Combining $L^- \leq c \leq L^+.$

⑤ If L^+ is finite then it is a limit point of $(a_n)_{n=m}^{\infty}$. Similarly, L^- is limit point if finite.

Proof: Let $\varepsilon > 0$ and $N \geq m$ be arbitrary.

define $a_M^+ = \sup (a_n)_{n=M}^{\infty}$ $\quad (M \geq m)$

Since $\lim_{M \to \infty} a_M^+ = L^+$

$\exists N_0$ s.t. $\forall K \geq N_0$

$|a_K^+ - L^+| \leq \varepsilon/2 \quad \forall \varepsilon > 0$

$\Rightarrow a_K^+ \leq L^+ + \varepsilon/2$

Define

$K = \max (M, N_0)$

$\Rightarrow K \geq N_0 \Rightarrow a_K^+ \leq L^+ + \varepsilon/2$

$\Rightarrow a_K^+ - \varepsilon/2 \leq L^+$

Since $a_K^+ = \sup (a_n)_{n \geq K}$

$\Rightarrow a_K^+ - \varepsilon/2$ can't be supremum.

$\Rightarrow \exists n \geq K$ s.t. $a_n > a_K^+ - \varepsilon/2$

$L^+ - \varepsilon/2 \leq a_K^+ < a_n + \varepsilon/2 \leq a_K^+ + \varepsilon/2$
$\leq L^+ + \varepsilon$
$< L^+ + 3\varepsilon/2$

$\Rightarrow L^+ - \varepsilon < a_n < L^+ + \varepsilon$

$\Rightarrow$ We have found $n \geq K \geq M$ s.t.

$|a_n - L^+| < \varepsilon$. This is true for all M & ε.

Same for the other case.

⑥ Let c be a real number. If $(a_n)_{n \geq m}$ converges to c then $L^+ = c = L^-$. Converse is true as well.

Proof: Let $\lim_{n \to \infty} a_n = c$ or

$\forall \varepsilon > 0 \quad \exists N$ s.t. $\forall n \geq N$

$|a_n - c| \leq \varepsilon.$

$\Rightarrow c - \varepsilon \leq a_n \leq c + \varepsilon.$

$\sup (a_n)_{n \geq N} \leq c + \varepsilon$ [supremum can be smaller than upper bound]

or $a_N^+ \leq c + \varepsilon$

similarly $\inf (a_n)_{n \geq N} \geq c - \varepsilon$ [infimum can be larger than lower bound]

$\Rightarrow L^+ \leq c, \; L^- \geq c$

$\Rightarrow L^- \geq c \geq L^+$ But $L^- \leq L^+$

$\Rightarrow L^+ = L^- \Rightarrow L^+ = L^- = c.$

$\Rightarrow$ Let $L^+ = L^- = c$

$\Rightarrow \lim_{N \to \infty} a_N^+ = c$

$\exists N_0$ s.t. $\forall N \geq N_0$

$|a_N^+ - c| \leq \varepsilon \quad \forall \varepsilon > 0.$

$\exists N_1$ s.t. $\forall N \geq N_1$

$|a_N^- - c| \leq \varepsilon \quad \forall \varepsilon > 0$

Take $N = \max \{N_0, N_1\}$

$c - \varepsilon \leq a_N^- \leq a_n \leq a_N^+ \leq c + \varepsilon \quad \forall n \geq N$

$\Rightarrow |a_n - c| \leq \varepsilon \quad \forall n \geq N$

$\Rightarrow (a_n)$ converge to c.

§ Lecture 12.1

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Claim: (a_n) (b_n) be two sequences of real numbers. Then $a_n \leq b_n \ \forall\ n \geq m$ implies:

$$\sup(a_n) \leq \sup(b_n)$$

$$\inf(a_n) \leq \inf(b_n)$$

$$\limsup(a_n) \leq \limsup(b_n)$$

$$\liminf(a_n) \leq \liminf(b_n).$$

Proof: 1 $\forall\ n \geq m$ $\quad a_n \leq b_n \leq \sup(b_n)$

$\Rightarrow$ $\sup(b_n)$ is an upper bound on (a_n)

$\Rightarrow$ $\sup(a_n) \leq \sup(b_n)$.

2 $\quad \inf(a_n) \leq a_n \leq b_n$

$\Rightarrow$ $\inf(a_n)$ is a lower bound on (b_n)

$\Rightarrow$ $\inf(a_n) \leq \inf(b_n)$ (But greatest lower bound is $\inf(b_n)$)

3 $\quad a_N^+ = \sup(a_n)_{n \geq N} \leq \sup(b_n)_{n \geq N} = b_N^+$

$$a_N^+ \leq b_N^+$$

$\Rightarrow$ $\inf a_N^+ \leq \inf b_N^+ \Leftrightarrow \limsup(a_n) \leq \limsup(b_n)$

4 similarly.

Claim: Squeeze test.

Let (a_n) (b_n) (c_n) be sequences of real numbers s.t.

$$a_n \leq b_n \leq c_n \quad \forall\ n \geq m.$$

If $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n$ then $\lim_{n\to\infty} c_n = L$.

Proof: A priori it is not clear if (c_n) is convergent. But if we can show that

L^+ of $(b_n) = L^-$ of (b_n) then

$$\lim_{n\to\infty} b_n = c.$$

$$a_n \leq b_n \leq c_n$$

$\Rightarrow$ $\liminf a_n \leq \liminf b_n \leq \liminf c_n$

$$L \leq \liminf(b_n) \leq L$$

$\Rightarrow$ $\liminf(b_n) = L$.

$$a_n \leq b_n \leq c_n$$

$\Rightarrow$ $\limsup(a_n) \leq \limsup(b_n) \leq \limsup(c_n)$

$$L \leq \limsup(b_n) \leq L$$

$$\limsup(b_n) = L$$

$\Rightarrow$ $\lim_{n\to\infty} b_n = L$.

EX: Let $a_n = \frac{(-1)^n}{n} + \frac{1}{n^2}$ $\quad n \geq 1$

$$-1+1 = 0$$

$$\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$$

$$-\frac{1}{3} + \frac{1}{9} = -2/9$$

$$\frac{1}{4} + \frac{1}{16} = 5/16$$

$$\frac{(-1)^n}{n} + \frac{1}{n^2} \leq \frac{1}{n} + \frac{1}{n}\cdot\frac{1}{n}$$

$$= \frac{1}{n}\left(\frac{1}{n}+1\right)$$

$$\leq \frac{2}{n}$$

$$\frac{(-1)^n}{n} + \frac{1}{n^2} \geq -\frac{1}{n} - \frac{1}{n} = -2/n$$

$\Rightarrow$ $-2/n \leq a_n \leq 2/n$

These two converge to 0

$\Rightarrow$ $\lim_{n\to\infty} a_n = 0$.

Ar. $\quad 2^{-n} \leq \frac{1}{n}$

For $n=1$ $\quad \frac{1}{2} \leq 1$

$$2^{-(n+1)} = \frac{1}{2}\, 2^{-n} \leq \frac{1}{2n}$$

$$\leq \frac{1}{n+1}$$

$\Rightarrow$ $0 \leq 2^{-n} \leq \frac{1}{n} \Rightarrow \lim_{n\to\infty} 2^{-n} = 0$.

Theorem: (Completeness of reals)

A sequence of real numbers is a Cauchy sequence if and only if it is convergent.

Proof: Every convergent sequence is Cauchy, we proved already.

Now every Cauchy sequence is convergent in reals.

Let (a_n) be a Cauchy sequence. Thus (a_n) is bounded.

$\Rightarrow$ L^+ and L^- are bounded.

If $L^+ = L^-$ then the sequence converges.

Let $\varepsilon > 0$ be any real number. Since (a_n) is Cauchy sequence $\Rightarrow$

$\exists\ N$ s.t. $\forall\ m, n \geq N$

$$|a_n - a_m| \leq \varepsilon \quad \forall\ \varepsilon > 0$$

or $a_N - \varepsilon \leq a_n \leq a_N + \varepsilon \quad \forall\ n \geq N$

$\Rightarrow$ $a_N - \varepsilon \leq \inf(a_n)_{n \geq k} \quad \forall\ k \geq N$

$\Rightarrow$ $\lim_{k\to\infty}(a_N - \varepsilon) \leq \liminf_{n\to\infty}(a_n)$

$\Rightarrow$ $L^- \geq a_N - \varepsilon \rightarrow$ 1

$$\sup(a_n)_{n \geq k} \leq a_N + \varepsilon$$

$\Rightarrow$ $\lim_{k\to\infty} \sup(a_n)_{n \geq k} \leq a_N + \varepsilon$

$$L^+ \leq a_N + \varepsilon \rightarrow 2$$

$$L^+ - L^- \leq a_N + \varepsilon - a_N + \varepsilon$$

$$= 2\varepsilon.$$

$$L^+ - L^- \geq 0 \geq -2\varepsilon$$

$\Rightarrow$ $|L^+ - L^-| \leq 2\varepsilon \quad \forall\ \varepsilon > 0$

$\Rightarrow$ $L^+ - L^- = 0 \Rightarrow L^+ = L^-$

$\Rightarrow$ sequence (a_n) converges.