§ Lecture 11.0

Monday, 15 September 2025 21:33

Consider the sequence :

$1.1, -1.01, 1.001, -1.0001, \cdots$

This sequence seems to be approaching 1 & -1.

But it is neither ε-close 1 or -1 for any fixed $\varepsilon > 0$.

These are not limits but something else.

Limit points:

Let $(a_n)_{n=m}^{\infty}$ be a sequence of real numbers.

Let $x \in \mathbb{R}$ and $\varepsilon > 0$.

(i) We say that x is ε-adherent to (a_n) iff there exists an $n \geq m$ s.t. a_n is ε-close to x.

(ii) We say that x is continually ε-adherent to (a_n) iff it is ε-adherent to $(a_n)_{n=N}^{\infty}$ $\forall N \geq m$.

$(a_m \; a_{m+1} \cdots\cdots)$
$(\quad a_{m+1} \cdots\cdots)$
$(\quad\quad a_{m+2} \cdots)$

$\Rightarrow$ You will keep on finding an index n s.t $|a_n - x| \leq \varepsilon$ no matter where you start.

(iii) x is called a limit point of $(a_n)_{n=m}^{\infty}$ iff it is continually ε-adherent $\forall \varepsilon > 0$.

i.e. x is called a limit point of $(a_n)_{n=m}^{\infty}$ iff

$\forall N \geq m \;\; \exists n \geq N$

$|a_n - x| \leq \varepsilon \;\; \forall \varepsilon > 0.$

(These are reversed.)

$1.1, -1.01, 1.001, -1.0001, \cdots$

Fix any ε, you will find $1.000\cdots1$

s.t. $|1 - 1.00\cdots1| = 0.0\cdots01 \leq \varepsilon$.

$\Rightarrow$ 1 is a limit point

Similarly -1 is a limit point.

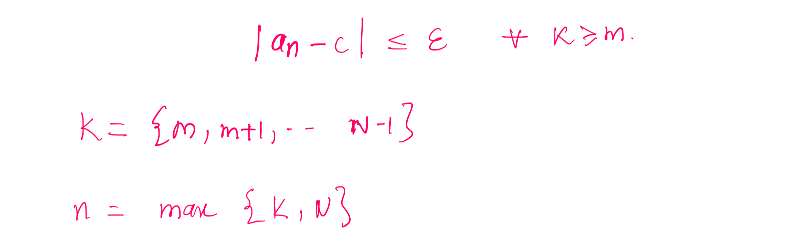
Is 0 a limit point? No.

Claim: (Limits are limit points)

Let $(a_n)_{n=m}^{\infty}$ be a sequence of reals that converges to c. Then c is a unique limit point of (a_n).

Proof: $\exists N \geq m$ s.t.

$|a_n - c| \leq \varepsilon \;\; \forall n \geq N$ & $\varepsilon > 0$.



No matter where you start $\exists \; n \geq K$ s.t.

$|a_n - c| \leq \varepsilon \;\; \forall K \geq m.$

$K = \{m, m+1, \cdots N-1\}$

$n = \max\{K, N\}$

$n \geq N. \Rightarrow |a_n - c| \leq \varepsilon$. This is true for all $K \geq m$.

$\Rightarrow c$ is a limit point.

Uniqueness:

Note that $\exists N \geq m$ s.t. $\forall n \geq N$

$|a_n - c| \leq \varepsilon \;\; \forall \varepsilon > 0$

Let $\varepsilon = \frac{|d-c|}{3}$ (when c, d are limit points)

Then $|a_n - c| \leq \frac{|d-c|}{3}$.

Since d is a limit point. $\Rightarrow$ For $K = N \; \exists n \geq N$

s.t. $|a_n - d| \leq \varepsilon \;\; \forall \varepsilon > 0.$
$= \frac{|d-c|}{3}$.

But $|d - c| = |a_n - c - a_n - d|$

$\leq 2\frac{|d-c|}{3}$ (Impossible.)

$\Rightarrow d = c.$

Special limit points.

Limit superior and limit inferior:

Suppose $(a_n)_{n=m}^{\infty}$ is a sequence.

$(a_m \;\; a_{m+1} \;\; a_{m+2} \;\; a_{m+3} \cdots\cdots)$
$(a_{m+1} \;\; a_{m+2} \;\; a_{m+3} \cdots)$
$(a_{m+2} \;\; a_{m+3} \cdots\cdots)$

$a_m^+ = \sup(\;)$ $\quad a_{m+1}^+ = \sup(\;)$.

$a_m^+ \geq a_{m+1}^+ \geq a_{m+2}^+ \cdots$ Decreasing sequence bounded by $\inf(a_n^+)_{n=m}^{\infty}$

$\Rightarrow \lim_{n\to\infty} a_n^+ = \inf(a_n^+)_{n=m}^{\infty}$

$a_N^+ = \sup(a_n)_{n=N}^{\infty}$ $\quad N = m, m+1 \cdots$

$\Rightarrow (a_N^+)_{N=m}^{\infty}$

$$\limsup_{n\to\infty} (a_n)_{n=m}^{\infty} = \inf (a_N^+)_{N=m}^{\infty}.$$

Similarly define $a_N^- = \inf(a_n)_{n=N}^{\infty}$

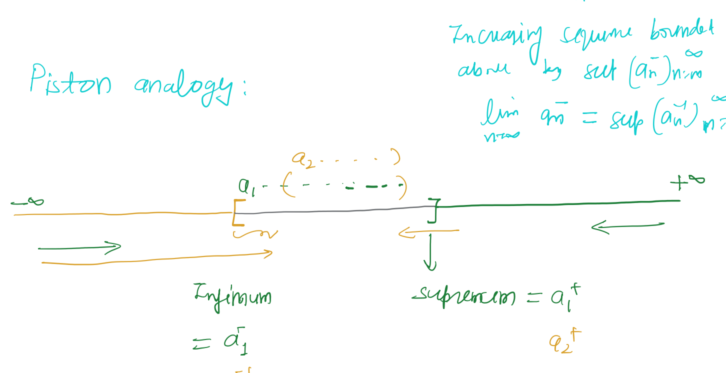
$$\liminf_{n\to\infty} (a_n)_{n=m}^{\infty} = \sup (a_N^-)_{N=m}^{\infty}.$$

$a_m^- \leq a_{m+1}^- \cdots$

Increasing sequence bounded above by $\sup(a_n)_{n=m}^{\infty}$

$\lim_{n\to\infty} a_n^- = \sup(a_n^-)_{n=m}^{\infty}$.

Piston analogy:



Consider a sequence

$$a_n = \frac{(-1)^{n-1}}{n} \qquad n = 1, 2, \cdots$$

Consider $a_N^+ = \sup(a_n)_{n=N}^{\infty}$

If N is even, then sequence looks like

$\left(-\frac{1}{N}, \frac{1}{N+1}, -\frac{1}{N+2}, \cdots\right)$

$\sup\left(-\frac{1}{N}, \frac{1}{N+1}, -\frac{1}{N+2}, \cdots\right) = \frac{1}{N+1}$ [Nothing can be smaller that is upper bound]

If N is odd then sequence looks like

$\left(\frac{1}{N}, -\frac{1}{N+1}, \cdots\right)$

$\sup(\quad) = \frac{1}{N}$.

Thus $a_N^+ = \begin{cases} \frac{1}{N} & \text{if } N \text{ is odd} \\ \frac{1}{N+1} & \text{if } N \text{ is even.} \end{cases}$

$\Rightarrow (a_N^+)_{N=1}^{\infty} = (1, 1/3, 1/3, 1/5, 1/5, \cdots)$

$\inf(1, 1/3, 1/3, 1/5, 1/5, \cdots)$

Since $a_N^+ \geq 0 \;\; \forall N = 1 \cdots$

0 is a lower bound.

Let $u > 0$ be another lower bound. Then

$a_N^+ \geq u \;\; \forall N = 1, 2, \cdots$ (This is not possible)

$\frac{1}{N}$ or $\frac{1}{N+1}$ (But we know that $\exists N > 0$ s.t. $\frac{1}{N+1} < \frac{1}{N} < u$)

$\Rightarrow$ 0 is infimum as there is no lower bound > 0.

Limit inferior is 0 too.

§ Lecture 11.1

Monday, 15 September 2025 22:22

Claim: Let $L^+ = \limsup_{n\to\infty} (a_n)_{n=m}^{\infty}$

$L^- = \liminf_{n\to\infty} (a_n)_{n=m}^{\infty}$

Then

(i) $\forall\ x > L^+\ \exists\ N \geq m$ s.t. $\forall\ n \geq N\quad a_n < x$.

$\forall\ y < L^-\ \exists\ N \geq m$ s.t. $\forall\ n \geq N\quad a_n > y$.

Proof: $x > \inf (a_N^+)_{N=m}^{\infty}$

$\Rightarrow$ There will exist some $N_0 \geq m$ s.t.

$x > a_{N_0}^+$

otherwise x will be a lower bound greater than infimum (contradicting definition of infimum)

$\Rightarrow\ x > a_{N_0}^+ = \sup (a_n)_{n=N_0}^{\infty} \geq a_n\quad \forall\ n \geq N_0$

$\Rightarrow\ x > a_n\quad \forall\ n \geq N_0$.

$y < L^- = \sup (a_N^-)_{N=m}^{\infty}$

$\Rightarrow\ y < a_{N_0}^-$ for some $N_0 \geq m$.

Otherwise $y \geq a_N\ \forall\ N \geq m$

$\Rightarrow$ y is an upperbound smaller than supremum. (contradiction)

$\Rightarrow\ y < \inf (a_n)_{n=N_0}^{\infty} \leq a_n\quad \forall\ n \geq N_0$

$\Rightarrow\ y < a_n\quad \forall\ n \geq N_0$ for some N_0.

Claim (ii):

$\inf (a_n) \leq L^- \leq L^+ \leq \sup (a_n)$

Proof: Fix any $N \geq m$.

$a_N^+ = \sup (a_n)_{n=N}^{\infty} \leq \sup (a_n)_{n=m}^{\infty}$

$a_N^- = \inf (a_n)_{n=N}^{\infty} \leq a_N^+$

$\inf (a_n)_{n=m}^{\infty} \leq a_N^- \leq a_N^+ \leq \sup (a_n)_{n=m}^{\infty}$

Taking limits we get

$\inf (a_n)_{n=m}^{\infty} \leq L^- \leq L^+ \leq \sup (a_n)_{n=m}^{\infty}$.

(iii) If c is a limit point then

$L^- \leq c \leq L^+$.

Proof: Since c is a limit point. Then for all

$N \geq m\quad \exists\ n_0 \geq N$ s.t. $|a_{n_0} - c| \leq \varepsilon\quad \forall\ \varepsilon > 0$.

$\Rightarrow\quad a_{n_0} \geq c - \varepsilon\quad \forall\ \varepsilon > 0$

$\Rightarrow$

$a_N^+ = \sup (a_n)_{n=N}^{\infty} \geq a_{n_0} \geq c - \varepsilon\quad \forall\ \varepsilon > 0$.

$\Rightarrow\ a_N^+ \geq c - \varepsilon\quad \forall\ \varepsilon > 0$

Taking infimum or limit

$L^+ \geq c - \varepsilon\ \forall\ \varepsilon > 0 \Rightarrow L^+ \geq c$.

$\Rightarrow\ a_N^- = \inf (a_n)_{n=N}^{\infty} \leq a_{n_0} \leq c + \varepsilon$

$\Rightarrow\ L^- \leq c + \varepsilon\quad \forall\ \varepsilon > 0$

$\Rightarrow\ L^- \leq c$.

$\Rightarrow\ L^- \leq c \leq L^+$.

(IV). $(a_n)_{n=m}^{\infty}$ converges to $c \Leftrightarrow L^+ = L^- = c$.

Proof: Let (a_n) converges to c. $\Rightarrow$

$\exists\ N_0 \geq m$ s.t. $\forall\ n \geq N$

$|a_n - c| \leq \varepsilon\quad \forall\ \varepsilon > 0$.

$\Rightarrow\quad c - \varepsilon \leq a_n \leq c + \varepsilon$

$a_N^+ = \sup (a_n)_{n=N}^{\infty} \leq c + \varepsilon\quad \forall\ \varepsilon > 0\ \forall\ N \geq N_0$

$a_N^- = \inf (a_n)_{n=N}^{\infty} \geq c - \varepsilon\quad \forall\ \varepsilon > 0\ \forall\ N \geq N_0$

$\hookrightarrow$ increasing sequence over N upper bounded by sup

Decreasing sequence over N lower bounded by inf.

Taking limits.

$\lim_{N\to\infty} a_N^+ \leq c + \varepsilon$

or $\inf a_N^+ \leq c + \varepsilon \Rightarrow L^+ \leq c + \varepsilon$

$\lim_{N\to\infty} a_N^- \geq c - \varepsilon$

$\sup a_N^- \geq c - \varepsilon \Rightarrow L^- \geq c - \varepsilon$.

$\Rightarrow$

$\Rightarrow\quad L^- \geq c\ ;\ L^+ \leq c$

$L^- \geq c \geq L^+ \Rightarrow L^- \geq L^+$

But $L^+ \geq L^-$

$\Rightarrow L^+ = L^-$

$\Rightarrow\ L \geq c \geq L$

$\Rightarrow L = c$.

$\Leftarrow\quad L^+ = L^- = c \Rightarrow a_n \to c$.

Proof: Since $a_N^+ \to c \Rightarrow \exists\ N_0$ s.t. $\forall\ N \geq N_0$

$\forall\quad |a_N^+ - c| \leq \varepsilon\quad \forall\ \varepsilon > 0$

or $c - \varepsilon \leq a_N^+ \leq c + \varepsilon$

Since $a_N^- \to c \Rightarrow \exists\ N_1$ s.t. $N \geq N_1$

$c - \varepsilon \leq a_N^- \leq c + \varepsilon\quad \forall\ \varepsilon > 0$.

Let $N = \max (N_0, N_1)$ For this N

$c - \varepsilon \leq a_N^- \leq a_n \leq a_N^+ \leq c + \varepsilon\quad \forall\ n \geq N\ \forall\ \varepsilon > 0$

$\Rightarrow\quad c - \varepsilon \leq a_n \leq c + \varepsilon$

or $|a_n - c| \leq \varepsilon$

$\Rightarrow\ a_n$ converges to c.

§ Lecture 11.2

Tuesday, 16 September 2025 00:16

Comparison law: Suppose $(a_n)_{n=m}^{\infty}$ and $(b_n)_{n=m}^{\infty}$ are two sequences of real numbers such that $a_n \le b_n \;\; \forall\, n \ge m$. Then

1. $\sup(a_n) \le \sup(b_n)$
2. $\inf(a_n) \le \inf(b_n)$
3. $\limsup(a_n) \le \limsup(b_n)$
4. $\liminf(a_n) \le \liminf(b_n)$

Proof: $A = \{a_n : n \ge m\}$

$B = \{b_n : n \ge m\}$

$a_N^+ = \sup\{a_n : n \ge N\}$ Same for b_N^+

$a_N^- = \inf\{a_n : n \ge N\}$ " " b_N^-.

(1) For every $n \ge m$ $a_n \le b_n \le \sup B$

$a_n \le \sup B \quad \forall\, n \ge m$

$\hookrightarrow$ upper bound

But $\sup(A)$ is a least upper bound on A

$\Rightarrow \sup(A) \le \sup(B)$

(2) $\forall\, n \ge m \quad a_n \le b_n$

$\inf(A) \le a_n \le b_n$

$\Rightarrow \inf(A)$ is a lower bound on B.

$\Rightarrow \inf(A) \le \inf(B)$

$\hookrightarrow$ greatest lower bound.

(3) Fix $N \ge m$. Because $a_n \le b_n \;\; \forall\, n \ge N$

$a_N^+ = \sup(a_n)_{n \ge N} \le \sup(b_n)_{n \ge N} = b_N^+$

$\Rightarrow a_N^+ \le b_N^+ \quad \forall\, N \ge m$

$\inf(a_N^+)_{N \ge m} \le \inf(b_N^+)_{N \ge m}$

$\limsup_{n\to\infty}(a_n) \le \limsup_{n\to\infty}(b_n)$

(4) For each $N \ge m$ $a_n \le b_n \;\; \forall\, n \ge N$

$a_N^- = \inf(a_n)_{n=N}^{\infty} \le \inf(b_n)_{n=N}^{\infty} \le b_N^-$

$\sup(a_N^-) \le \sup b_N^-$.

Squeeze test: $(a_n)_{n=m}^{\infty}$ $(b_n)_{n=m}^{\infty}$ $(c_n)_{n=m}^{\infty}$ s.t.

$a_n \le b_n \le c_n \quad \forall\, n \ge m.$

suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

Proof: $a_n \le b_n \quad \forall\, n \ge m$ $\longrightarrow$ $\lim b_n$ is not known to exist!!

$\Rightarrow \liminf(a_n) \le \liminf(b_n)$

$b_n \le c_n$

$\Rightarrow \limsup(b_n) \le \limsup(c_n)$

But $\liminf(b_n) \le \limsup(b_n)$

$\Rightarrow L \le \liminf(b_n) \le \limsup(b_n) \le L$

$\Rightarrow \limsup(b_n) = L = \liminf(b_n)$

$\Rightarrow \lim_{n\to\infty} b_n = L.$